1 State-Space Systems

1.1 State Space Descriptions of Dynamical Systems

The *states* are a set of system variables that represent the memory of a (causal) dynamical system. A dynamical system's state $\underline{x}$ must satisfy the following two properties:

1. For any $t_o < t_1, \underline{x}(t_1)$ can always be determined from $\underline{x}(t_o)$ and $\{\underline{u}(\tau), t_o \le \tau \le t_1\}$.
2. The outputs y at time t, is a *memoryless* function of $\underline{x}(t)$ and $\underline{u}(t)$.

1.2 Linearizing Nonlinear Dynamical Systems

Suppose we have a nonlinear (time invariant) dynamical system in state space form:

$$\mathcal{S}\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) \\ \underline{y} = \underline{g}(\underline{x}) \end{cases}$$

Consider small perturbations from an equilibrium state $\underline{x}_e$ when $\underline{x}(t) = \underline{u}_e$:

$$\delta\dot{\underline{x}} \cong A\delta\underline{x} + B\delta\underline{u}$$

$A = \frac{\partial f}{\partial \underline{x}}(\underline{x}_e, \underline{u}_e)$ and $B = \frac{\partial f}{\partial \underline{u}}(\underline{x}_e, \underline{u}_e)$:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\underline{x}=\underline{x}_e, \underline{u}=\underline{u}_e}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\underline{x}=\underline{x}_e, \underline{u}=\underline{u}_e}$$

Similarly, $\delta\underline{y} = C\delta\underline{x}$ where $C = \frac{\partial \underline{g}(\underline{x}_e)}{\partial \underline{x}}$.

For linear time-invariant dynamical systems, the standard form is:

$$\mathcal{S}\begin{cases} \dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \end{cases}$$

For *implicit* state equations, $\underline{F}(\dot{\underline{x}}, \underline{x}, \underline{u}) = \underline{0}$. The linearized model can be derived from the equilibrium $\underline{F}(\underline{0}, \underline{x}_e, \underline{u}_e) = \underline{0}$:

$$L\delta\dot{\underline{x}} + M\delta\underline{x} + N\delta\underline{u} \simeq \underline{0}$$

$$\delta\dot{\underline{x}} \simeq -L^{-1}M\delta\underline{x} - L^{-1}N\delta\underline{u}$$

$L = \frac{\partial F}{\partial \dot{\underline{x}}}(\underline{0}, \underline{x}_e, \underline{u}_e)$, $M = \frac{\partial F}{\partial \underline{x}}(\underline{0}, \underline{x}_e, \underline{u}_e)$ and $N = \frac{\partial F}{\partial \underline{u}}(\underline{0}, \underline{x}_e, \underline{u}_e)$.

1.3 State-space Trajectories

Consider the free motion of the time invariant dynamical system $\dot{\underline{x}} = \underline{f}(\underline{x})$:

$$\underline{x}(t + \delta t) \cong \underline{x}(t) + \underline{f}(\underline{x}(t))\delta t$$

For a linear time-invariant dynamical system $\dot{\underline{x}} = A\underline{x}$ where A has eigenvalues λ_i with corresponding eigenvectors $\underline{w}_i$. Suppose $\underline{x} = k\underline{w}_1$:

$$\dot{\underline{x}} = \lambda_1\underline{x} \Rightarrow \underline{x}(t) = e^{\lambda_1 t}\underline{x}_0$$

Velocity fields in the state-space:

1. Complex eigenvalue with positive real part $\Rightarrow$ spiral out.
2. Complex eigenvalue with negative real part $\Rightarrow$ spiral in.
3. Positive eigenvalue $\Rightarrow$ unstable manifolds.
4. Negative eigenvalue $\Rightarrow$ stable manifolds.

Given the complex conjugate roots $\lambda = \rho \pm i\sigma$:

$$\zeta = -\frac{\rho}{\sqrt{\rho^2 + \sigma^2}}$$

Rule of thumb: Number of cycles to half-amplitude $\simeq \frac{11}{\zeta}$.

1.4 Solutions to Linear State Equations

Taking Laplace Transform of the linear time-invariant dynamical system:

$$\underline{Y}(s) = C(sI - A)^{-1}\underline{x}_0 + \left(D + C(sI - A)^{-1}B\right)\underline{U}(s)$$

First term is the initial condition response and second term is the input response. For $\underline{x}_0 = \underline{0}$:

$$\underline{Y}(s) = \left(D + C(sI - A)^{-1}B\right)\underline{U}(s)$$

$G(s) = D + C(sI - A)^{-1}B$ is called the *transfer function matrix*. Poles of $G(s)$ are values of B at which the transfer function becomes infinite:

$$\det(sI - A) = 0$$

Poles of $G(s) \subseteq$ eigenvalues of A.

For $\underline{u}(t) = \underline{0}$, $\underline{X}(s) = (sI - A)^{-1}\underline{x}_0$:

$$\underline{x}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\underline{x}_0 = \Phi(t)\underline{x}_0$$

$\Phi(t)$ is called the *state transition matrix*:

$$\Phi(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \overset{\text{def}}{=} e^{At}$$

Change of state coordinates: If $A = T^{-1}\overline{A}T$ then $e^{At} = T^{-1}e^{\overline{A}t}T$.

For *non-defective* A we have the eigenvalue/eigenvector decomposition:

$$A = W\Lambda W^{-1}$$

So that for $T = W^{-1}$ we have $\Lambda = \text{diag}\,\{\lambda_i\}$ and $e^{\Lambda t} = \text{diag}\left\{e^{\lambda_i t}\right\}$.

Semigroup property:

$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2} = e^{At_2}e^{At_1}$$

Inverse:

$$\left(e^{At}\right)^{-1} = e^{-At}$$

Derivative:

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

Integral:

$$\int_0^t e^{A\tau}d\tau = A^{-1}e^{At} - A^{-1} \qquad \text{if } \det(A) \neq 0$$

1.5 Convolution Integral

Consider $\dot{\underline{x}} - Ax = B\underline{u}$:

$$\underline{y}(t) = Ce^{At}\underline{x}_0 + D\underline{u}(t) + \int_0^t Ce^{A(t-\tau)}B\underline{u}(\tau)d\tau$$

For $\underline{x}_0 = \underline{0}$:

$$y(t) = \int_0^t H(t - \tau)\underline{u}(\tau)d\tau = H(t) * \underline{u}(t)$$

$H(t)$ is called the *impulse response matrix*:

$$H(t) = \begin{cases} D\delta(t) + Ce^{At}B & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Note that the transfer function, $G(s) = \mathcal{L}(H(t))$.

1.6 Frequency Response

Consider a linear time-invariant system that is *asymptotically stable*. For a sinusoidal input $u_j(t) = A_j\cos\left(\omega_o t + \theta_j\right)$:

$$\lim_{t\to\infty} y_i(t) = B_i\cos(\omega_o t + \phi_i)$$

$$B_i e^{j\phi_i} = \sum_{j=1}^{m} g_{ij}(j\omega_o)A_j e^{j\theta_j}$$

1.7 State Space Equations for Composite Systems

Cascade of two systems: $\underline{Y}(s) = G_2(s)G_1(s)\underline{U}(s)$ is realized by:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}\underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}\underline{x}(t) + D_2D_1\underline{u}(t)$$

Parallel combination of two systems: $\underline{Y}(s) = (G_1(s) + G_2(s))\,\underline{U}(s)$ is realized by:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\underline{u}(t)$$

$$\underline{y}(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\underline{x}(t) + (D_1 + D_2)\underline{u}(t)$$

2 Classical Methods

2.1 Feedback Control

Let $L(s)$ be the (open) loop transfer function (return-ratio): $L(s) = G(s)K(s)$

Complementary Sensitivity:

$$T(s) = \frac{L(s)}{1 + L(s)}$$

$$\overline{y} = T\overline{r} = -T\overline{n}$$

$\overline{r}$ is the reference and $\overline{n}$ is the sensor noise.

Sensitivity:

$$S(s) = \frac{1}{1 + L(s)}$$

$$\overline{e} = S\overline{r}, \quad S = \frac{dT/T}{dG/G}, \quad \overline{y} = GS\overline{d}$$

$\overline{e}$ is the error and $\overline{d}$ is the disturbance.

Trade-off between S and T:

$$S(s) + T(s) = 1$$

2.2 The Root-Locus Method

Root-locus diagram shows the locations of the roots of $1 + kL(s) = 0$ for $k > 0$.

$$L(s) = \frac{n(s)}{d(s)} = \frac{c(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

Rules for constructing the root-locus plot:

1. The root-locus diagram is symmetric with respect to the real axis and consists of n branches.
2. For $k = 0$ the n branches start at the open loop poles p_i. As $k \to \infty, m$ branches tend to the zeros z_i and $n - m$ branches tend to infinity.
3. Points on the real axis which lie to the left of an odd number of poles and zeros are on the root-locus.
4. The breakaway points are those points on the root-locus for which $\frac{d}{ds}L(s) = 0$(same as $dk/ds = 0$).
5. As $k \to \infty$, the $n - m$ branches which tend to infinity do so along straight line asymptotes at angles $(2\ell + 1)\pi/(n - m)$ to the +ve real axis $(\ell = 0, \ldots, n - m - 1)$, and emanate from the point:

$$\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}$$

The *angle condition* for s_0 to be on the root-locus:

$$\angle L(s_0) = \sum_{i=1}^{m}\angle(s_0 - z_i) - \sum_{i=1}^{n} L(s_0 - p_i) = (2\ell + 1)\pi$$

At a point s_0 on the root-locus:

$$k = \frac{1}{|L(s_0)|} = \frac{1}{c}\frac{\prod_{i=1}^{n}|s_0 - p_i|}{\prod_{i=1}^{m}|s_0 - z_i|}$$

2.3 The Routh-Hurwitz Criterion

The closed-loop characteristic equation $1 + G(s)K(s) = 0$ has the same roots as $d_G(s)d_K(s) + n_G(s)n_K(s) = 0$.

The Routh-Hurwitz criterion: Consider the polynomial (assume $a_0 > 0$):

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_{n-1}s + a_n$$

All roots have negative real parts only if $a_i > 0$ for each i. A *Routh array* can be constructed for arbitrary n.

3 Observability and Observers

3.1 Observability

A system is called *observable* if we can deduce the state, $\underline{x}(t)$, from measurements of $\underline{u}(\tau)$ and $\underline{y}(\tau)$ over some time interval (t_1, t_2) with $t_1 < t < t_2$.

Consider differentiating $\underline{y}(t)$ to give:

$$\begin{bmatrix} \underline{y}(t) \\ \dot{\underline{y}}(t) \\ \ddot{\underline{y}}(t) \\ \vdots \\ \underline{y}^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ CB\underline{u}(t) \\ CAB\underline{u}(t) + CB\dot{\underline{u}}(t) \\ \vdots \\ CA^{n-2}B\underline{u} + \cdots + CB\underline{u}^{(n-2)} \end{bmatrix}$$

Defining the *observability matrix*:

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The Observability test: The system is observable if and only if rank $Q = n$.

If a system is not observable, there will exist a vector $\underline{x}_o \neq 0$ for which $Q\underline{x}_o = 0$. This is called an *unobservable state*.

Observability Gramian:

$$W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$$

$W_o(t_1)$ is a positive semi-definite matrix. The system will be observable if $\underline{d}^T W_o(t_1)\underline{d} > 0$ for all $\underline{d} \neq \underline{0}$, i.e. if $W_o(t_1)$ is a positive definite matrix.

$$\int_0^{t_1} \left\| \underline{y}(t) - \underline{y}_o(t) \right\|^2 dt = \underline{d}^T W_o(t_1)\underline{d}$$

3.2 Observers

Use a *state observer* (Luenberger Observer) whose state $\hat{x}(t)$, approaches $\underline{x}(t)$ as $t \to \infty$:

$$\begin{cases} \dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - \hat{\underline{y}}) \\ \hat{\underline{y}} = C\hat{\underline{x}} \end{cases}$$

Consider the error $\underline{e}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$:

$$\dot{\underline{e}} = \dot{\underline{x}} - \dot{\hat{\underline{x}}} = (A - LC)\underline{e}$$

If the eigenvalues of $(A - LC)$ are large and negative, then $e^{(A-LC)t} \to 0$ quickly as t increases. We can arbitrarily assign the eigenvalues of $(A - LC)$ by choice of L if and only if the system is observable.

Consider random turbulence $d(t)$ and measurement with noise $n(t)$

$$\begin{cases} \dot{\underline{x}} = A\underline{x}(t) + Bd(t) \\ y(t) = C\underline{x}(t) + n(t) \end{cases}$$

Use the observer $\dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}(t) + L[y(t) - C\hat{\underline{x}}(t)]$ for tracking disturbances and ignoring noise:

1. If d large, n small: Believe the measurements and use large L.
2. If d small, n large: Believe model and use small L.

Sensor bias estimation: For sensor measurement with bias $y = \dot{\theta} + b_\omega$, augment state vector $\underline{x} = \left[\theta, \dot{\theta}, b_\omega\right]^T$ and assume bias is constant $\dot{b}_\omega = 0$.

Disturbance observer: Consider the augmented state-space model:

$$\begin{cases} \dot{d} = 0 \\ \dot{x} = Ax + B(u + d) \\ y = Cx \end{cases}$$

The observability matrix of the augmented model is:

$$\tilde{Q} = \begin{bmatrix} 0 & C \\ QB & QA \end{bmatrix}$$

$$\text{rank}\,\tilde{Q} = \text{rank}\begin{bmatrix} -CA^{-1}B & C \\ 0 & QA \end{bmatrix} = n + m$$

The internal model principle: A disturbance modeled by $\ddot{d} + \omega_0^2 d = 0$ is observable iff $G(j\omega_0)$ has rank m.

3.3 Observable and Unobservable Subspaces

QR factorization: If the rank $Q = r < n$ then there exists a nonsingular $n \times n$ matrix T and a $n \times r$ matrix $\tilde{Q}_1$ of rank r, such that $Q = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix} T$.

Change the state coordinates to $\tilde{\underline{x}} = T\underline{x}$:

$$\begin{cases} \dot{\tilde{\underline{x}}} = TAT^{-1}\tilde{\underline{x}} + TB\underline{u} = \tilde{A}\tilde{\underline{x}} + \tilde{B}\underline{u} \\ \underline{y} = CT^{-1}\tilde{\underline{x}} = \tilde{C}\tilde{\underline{x}} \end{cases}$$

In these coordinates if we partition the state, $\tilde{\underline{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$ with $\tilde{x}_1$ of dimension r and compatibly partition:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix};$$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$$

Then $\tilde{C}_2 = 0$, $\tilde{A}_{12} = 0$, and $\left(\tilde{A}_{11}, \tilde{C}_1\right)$ is observable. In these state coordinates we have $\dot{\tilde{x}}_1 = \tilde{A}_{11}\tilde{\underline{x}}_1 + \tilde{B}_1\underline{u}$, $\underline{y} = \tilde{C}_1\tilde{\underline{x}}_1$ and the input/output response (i.e. the transfer function) depends only on $\tilde{x}_1$ and the states $\tilde{x}_2$ are all unobservable.

Subspace: Let $T^{-1} = [X\,Y]$, Y in $\mathbb{R}^{n\times r}$ is a basis for null(Q), which is called $\overline{O}$, the unobservable subspace and X complements Y.

4 Controllability and State Feedback

4.1 Controllability

A state-space model is controllable if given an initial state $\underline{x}_0$ and a final state $\underline{x}_f$, there exists an input signal $u(\cdot)$ that transfers the state from $\underline{x}(0) = \underline{x}_0$ to $\underline{x}(T) = \underline{x}_f$ in a finite time $T > 0$.

For discrete-time model: $\underline{x}[k+1] = A\underline{x}[k] + B\underline{u}[k]$

$$\underline{x}_f - A^n\underline{x}_0 = \begin{bmatrix} B \,|\, & \ldots & |\, A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ \vdots \\ \underline{u}[0] \end{bmatrix}$$

Defining the *controllability matrix*:

$$P = \begin{bmatrix} B \,|\, & \ldots & |\, A^{n-1}B \end{bmatrix}$$

A solution exists if the controllability matrix P has full rank n.

Controllability Gramian:

$$W_c(t_1) = \int_0^{t_1} e^{A\tau}BB^T e^{A^T\tau} d\tau$$

If $\det W_c(t_1) \neq 0$ then we can reach any $\underline{x}(t_1)$ from $\underline{x}(0) = \underline{0}$ using $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1 - t)} W_c(t_1)^{-1}\underline{x}_1$.

$$\int_0^{t_1} \underline{u}_o(t)^T \underline{u}_o(t) dt = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1$$

The inverse of the controllability Gramian defines the input energy required to transfer the state from $x_0 = 0$ to x_1.

If $W_c(t_1)$ is a singular matrix there exists $\underline{z} \neq \underline{0}$ such that $\underline{z}^T W_c(t_1)\underline{z} = 0$:

$$\int_0^{t_1} \left(\underline{z}^T e^{A\tau} B\right)\left(B^T e^{A^T\tau}\underline{z}\right) d\tau = 0$$

Hence z is in Null space of P^T = Null space of $W_c(t_1)$. The null space of P^T defines the *unreachable subspace*.

The input, $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1 - t)} W_c(t_1)^{-1}\underline{x}_1$, takes the state from $\underline{x}(0) = \underline{0}$ to $\underline{x}(t_1) = \underline{x}_1$ with *minimum energy*.

4.2 State Feedback Design

The system: $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, with state feedback: $\underline{u} = -K\underline{x} + M\underline{r}$, giving closed loop:

$$\dot{\underline{x}} = (A - BK)\underline{x} + BM\underline{r}$$

The closed loop poles will be the eigenvalues of $(A - BK)$ which can be placed arbitrarily by choice of K if and only if (A, B) is controllable.

In steady-state: $\dot{\underline{x}} = \underline{0}$

$$\underline{y} = C\underline{x} = C(-A + BK)^{-1}BM\underline{r}$$

Choose M such that $C(-A+BK)^{-1}BM = I$ and $\underline{y}(t) \to \underline{r}$ after a step change with speed given by eigenvalues of $(A - BK)$.

Integral action: Augment the state by the integral of the error $\dot{\underline{e}} = \underline{r} - \underline{y} = \underline{r} - C\underline{x}$ with state feedback:

$$\underline{u} = -K_1\underline{x} - K_2\underline{e}$$

Choose K_1, K_2 to assign the closed-loop poles (possible if augmented system controllable) and then $\underline{e}(t) \to 0 \Rightarrow \underline{y}(t) \to \underline{r}$ after a step change.

4.3 Subspaces and Minimal Realizations

(A, B) controllable iff $\left(A^T, B^T\right)$ observable. Hence any statement about observability can be turned into a statement about controllability and vice-versa.

If rank $Q = r < n$, then there exists a decomposition $\tilde{\underline{x}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$ such that:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}; \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}$$

$\left(\tilde{A}_{11}, \tilde{C}_1\right)$ defines an observable subsystem. The state $\tilde{\underline{x}}_2$ does not affect the output and is unobservable.

If rank $P = r < n$, the corresponding decomposition leads to:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

The subsystem $\left(\tilde{A}_{11}, \tilde{B}_1\right)$ is controllable. The state $\underline{x}_1$ lives in the reachable subspace. The state $\underline{x}_2$ is uncontrollable and its trajectory is not affected by the input.

Kalman decomposition:

$$\tilde{\underline{x}} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 \end{bmatrix}^T$$

$\tilde{x}_1$ and $\tilde{x}_2$ are controllable while $\tilde{x}_2$ and $\tilde{x}_3$ are observable.

$$\dot{\underline{x}} = \left[\begin{array}{cc|cc} A_{11} & \star & \star & \star \\ 0 & A_{22} & 0 & \star \\ \hline & 0 & A_{33} & \star \\ & & 0 & A_{44} \end{array}\right]\underline{x} + \begin{bmatrix} B_1 & B_2 & 0 & 0 \end{bmatrix}^T \underline{u}$$

$$\underline{y} = \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix}\underline{x}$$

$H(s) = C_2(sI - A_{22})^{-1}B_2$ only depends on the states that are both observable and controllable.

A set of state equations given by (A, B, C, D) is called a *minimal realization* of its transfer function, $G(s) = D + C(sI - A)^{-1}B$, if there does not exist a state space realization of $G(s)$ with a lower state dimension. A realization is minimal if and only if it is both controllable and observable.

$$\begin{cases} \dot{\tilde{x}}_2 = A_{22}\tilde{x}_2 + B_2 u \\ y = C_2\tilde{x}_2 \end{cases}$$

For a single-input/single-output (SISO) transfer function, non-minimal realization implies *pole-zero cancellation*.

5 Combined System Control

5.1 Observers with State Feedback

We can combine the observer and the state feedback to control a system with $\underline{u} = -K\hat{\underline{x}} + M\underline{r}$. Consider the error $\underline{e} = \underline{x} - \hat{\underline{x}}$:

$$\dot{\underline{e}} = (A - LC)\underline{e}$$
$$\underline{u} = -K(\underline{x} - \underline{e}) + M\underline{r}$$
$$\dot{\underline{x}} = (A - BK)\underline{x} + BK\underline{e} + BM\underline{r}$$

Hence the closed loop system:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} BM \\ 0 \end{bmatrix}\underline{r}$$

$$\underline{y} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix}$$

Eigenvalues of $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \{$ eigenvalues of $X\} \cup \{$ eigenvalues of $Z\}$. So closed-loop poles are at the eigenvalues of $(A - BK)$ and those of $(A - LC)$. e is not affected by $\underline{r}$ so that $\underline{e}(t) \to \underline{0}$.

5.2 The Linear Quadratic Regulator

Consider the system $\underline{x}(0) = \underline{x}_o$ and the problem of choosing $\underline{u}$ to minimise $\int_0^\infty \underline{y}^T \underline{y} + \underline{u}^T R \underline{u} dt$ for some positive definite (weight) matrix $R = R^T > 0$.

The Control Algebraic Ricatti Equation (CARE):

$$XA + A^T X + C^T C - XBR^{-1} B^T X = 0$$

If the the system is controllable and observable, then the equation has a unique positive definite solution $X = X^T > 0$.

The minimizing control is the state feedback $\underline{u} = -K\underline{x}$ with $K = R^{-1} B^T X$.

The weight matrix R determines the relative penalty over the input and the output energy.

1. *Cheap control:* choose $R = rI$ and let $r \to 0$
2. *Minimum energy control:* choose $R = rI$ and let $r \to \infty$.

R is often chosen diagonal. A different scaling factor can be applied to different actuators.

5.3 Kalman Filter

(The End)