1 Continuous Time Systems

1.1 Continuous Time Signals and Systems

A *signal* is real-valued function $x(t)$, $0 \le t < \infty$. A *system* turns input signals, $u(t)$ into outputs, $y(t)$. Signals may be smooth and/or continuous. One way to represent a (smooth) system is as a linear ODE.

The *Laplace transform* for a continuous time signal $x(t)$:

$$\overline{x}(s) = \int_{0^+}^{\infty} x(t)e^{-st}dt$$

The *Transfer Function* of the system is:

$$G(s) = \frac{\overline{y}(s)}{\overline{u}(s)}$$

We can also represent the system using a convolution representation, using the system's impulse response, $g(t)$:

$$y(t) = \int_0^{\infty} g(t-\tau)u(\tau)d\tau$$

Taking Laplace transforms gives $\overline{y}(s) = \overline{g}(s)\overline{u}(s)$.

A system, L, is *Linear time-invariant* (LTI), iff the following conditions both hold:

1. *Linear:* If $L(u_1(t)) = y_1(t)$ and $L(u_2(t)) = y_2(t)$, then for any scalars α_1, α_2:

$$\begin{aligned} L(\alpha_1 u_1(t) + \alpha_2 u_2(t)) \\ = \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

2. *Time-invariant:* If $L(u(t)) = y(t)$ then $L(u(t+T)) = y(t+T)$ for any time interval, T, i.e., the system's response properties do not change over time.

A system is *stable* if "bounded inputs give bounded outputs." For a system with a rational transfer function, $G(s)$, this means G has no poles in the right-half complex plane or imaginary axis.

For a stable LTI system, $G(s)$, the *steady-state response* for an input $u(t) = \sin(\omega t)$ is:

$$y_{ss}(t) = |G(j\omega)|\sin(\omega t + \angle(G(j\omega)))$$

The *Nyquist Stability Criterion* allows us to determine the stability of a closed loop system by analysing its open loop properties:

1. Plot the path in the complex plane of $H(j\omega)G(j\omega)$ for $-\infty < \omega < \infty$.
2. Let N = number of anti-clockwise encirclements of the point -1.
3. Then the closed loop system is stable if and only if N = number of RHP poles of $H(s)G(s)$.

2 Discrete Time Systems

2.1 Discrete Time Signals and Systems

A *discrete time signal* is a number sequence, indexed from zero: $\{x_k\}_{k\ge 0}$ or $\{x(kT)\}_{k\ge 0}$, where T is the *sampling period*. A *discrete time system* takes discrete time signal inputs and produces discrete time signal outputs.

2.2 The Z Transform

The *Z transform* of the signal $\{x_k\}_{k\ge 0}$ is defined as:

$$\overline{x}(z) = \mathcal{Z}[x_k] = \sum_{k=0}^{\infty} x_k z^{-k}$$

Properties of the Z transform: Let $\{x_k\}, \{y_k\}$ be discrete time signals whose Z transforms exist:

1. *Linearity:* For any scalars α, β:

$$\begin{aligned} &\mathcal{Z}[\alpha\{x_k\} + \beta\{y_k\}] \\ &\quad = \alpha\mathcal{Z}[\{x_k\}] + \beta\mathcal{Z}[\{y_k\}] \end{aligned}$$

2. *Time delay:* Define the time delay operation $\{x_k\} \mapsto \{x_{k-1}\}$.

$$\mathcal{Z}[\{x_{k-1}\}] = x_{-1} + z^{-1}\overline{x}(z)$$

z^{-1} is the *time-delay operator*.

3. *Time advance:* $\{x_k\} \mapsto \{x_{k+1}\}$.

$$\mathcal{Z}[\{x_{k+1}\}] = -zX_0 + z\overline{X}(z)$$

z is the *time-advance operator*.

4. *Scaling:*

$$\mathcal{Z}\left[\left\{r^k x_k\right\}\right] = \overline{x}\left(r^{-1}z\right)$$

5. *Initial Value Theorem:*

$$\lim_{z\to\infty} \overline{x}(z) = x_0$$

6. *Convolution:*

$$\{x_k\} * \{y_k\} = \sum_{i=0}^{k} x_i y_{k-i} = \sum_{i=0}^{k} x_{k-i} y_i$$

$$\mathcal{Z}[\{x_k\} * \{y_k\}] = \overline{x}(z)\overline{y}(z)$$

Inversion of the Z transform: The inverse of the Z transform can be computed by manipulating expressions in the z-domain and identify *standard transforms*.

2.3 Z-Transfer Function

A System described by linear difference equations $y_k + a_1 y_{k-1} + \ldots + a_n y_{k-n} = b_0 u_k + \ldots + b_m u_{k-m}$ and subject to zero initial conditions $y_k = u_k = 0$ for $k < 0$ is *linear* and *time-invariant*. Such a system has a *z-transfer function*:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

2.4 Pulse Response of LTI Systems

The *unit pulse* signal, δ_k in discrete time is defined as:

$$\delta_k = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider the unit pulse input, $\{u_k\}_{k\ge 0} = \delta_k = (1, 0, 0, 0, \ldots)$ to a system. The output of the system $\{g_k\}_{k\ge 0} = (g_0, g_1, g_2, g_3, \ldots)$ is called the *pulse response* of the system.

Any discrete-time LTI system can be represented as a convolution. Take a general input $\{u_k\}_{k\ge 0} = (u_0, u_1, u_2, u_3, \ldots)$, the output $\{y_k\}_{k\ge 0}$ can be computed as:

$$y_k = \sum_{i=0}^{k} u_i g_{k-i} = \sum_{i=0}^{k} u_{k-i} g_i$$
$$\{y_k\} = \{g_k\} * \{u_k\}$$

Taking Z transforms gives $\overline{y}(z) = \overline{g}(z)\overline{u}(z)$. The transfer function equals the Z transform of the pulse response.

3 Digital Filters

3.1 FIR, IIR and Causality

Digital filters whose pulse response terminates after a finite number of time steps are called *Finite Impulse Response (FIR)* filters/systems.

$$\{g_k\} = (g_0, g_1, g_2, \ldots g_n, 0, 0, \ldots, 0)$$

Otherwise, the system is called Infinite Impulse Response (IIR).

FIR systems have transfer functions with a special form:

$$\begin{aligned} G(z) &= g_0 + g_1 z^{-1} + \ldots + g_n z^{-n} \\ &= \frac{z^n g_0 + \ldots + g_n}{z^n} \end{aligned}$$

All of the poles of the transfer function of an FIR filter are at $z = 0$. By contrast, IIR filters can have poles at arbitrary locations.

Discrete time systems whose pulse response is zero for negative time are called *causal*. The transfer functions of causal systems have the form:

$$G(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \ldots$$

For causal systems, $G(z)$ is finite as $z \to \infty$.

3.2 Stability

We say that a signal $\{u_k\}$ is bounded if there exists a positive constant M such that $|u_k| < M$ for all k. A discrete time system is *stable* if bounded inputs give bounded outputs (*BIBO stability*).

Let G be a discrete time system with a rational transfer function:

$$G(z) = \frac{b_0 z^m + b_1 z^{m-1} + \ldots + b_m}{z^n + a_1 z^{n-1} + \ldots + a_n}$$

For the system to be causal, we must have $m \le n$.

Conditions for stability of a discrete time system: Let the pulse response of G be $\{g_k\}_{k\ge 0}$. Then the following are equivalent:

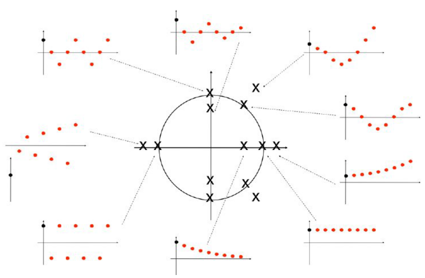
1. G is stable
2. All of the roots, p_i, of $d(z)$ satisfy $|p_i| < 1$.
3. $\sum_{k=0}^{\infty} |g_k|$ is finite.

Suppose $p_1, p_2, p_3, \ldots$ are distinct. Then we can decompose G using partial fractions:

$$G(z) = \frac{\alpha_1}{1 - p_1 z^{-1}} + \ldots + \frac{\alpha_n}{1 - p_n z^{-1}}$$

Then $g_k = \alpha_1 p_1^k + \ldots + \alpha_n p_n^k$.

Poles of G(z) define the response to a pulse:



In the z-domain any linear filter can be written as:

$$A(z)Y(z) = B(z)U(z) + C(z, y_i)$$

$C(z, y_i)$ takes into account the *initial conditions* of the filter.

$$\lim_{k\to\infty} y(k) = \lim_{k\to\infty} \mathcal{Z}^{-1}\left[\frac{B(z)}{A(z)}U(z)\right]$$

Stable roots in $A(z)$ enforce the exponential decay of $\mathcal{Z}^{-1}\left[\frac{C(z,y_i)}{A(z)}\right]$.

Final Value Theorem: Suppose that all the poles of $(z-1)Y(z)$ lie strictly inside the unit circle.

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1} (z-1)Y(z)$$

Steady-state response to a unit step input $U(z) = \frac{z}{z-1}$:

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1} zG(z) = G(1)$$

4 Frequency Response

4.1 Frequency Response

For digital filters, its frequency response can be characterized by its response to a sinusoidal input. For a sampling time of T and sampled input signal $\cos(k\theta)$ for $\theta = \omega T$, the output signal $y(k)$ at steady state is:

$$y_{ss}(k) = \left|G\left(e^{j\theta}\right)\right|\cos\left(\theta k + \angle G\left(e^{j\theta}\right)\right)$$

The filter amplifies the sinusoidal input $\cos(\theta k)$ by a factor $\left|G\left(e^{j\theta}\right)\right|$ and shifts its phase by a factor $\angle G\left(e^{j\theta}\right)$.

By Shannon sampling theorem the sampling frequency $f \ge 2\omega_{\max}$, which gives a sampling time $T = \frac{2\pi}{f} \le \frac{\pi}{\omega_{\max}}$. Thus, $\theta = \omega T \le \pi$.

4.2 Bode Diagram

Let G be a digital filter with transfer function:

$$G(z) = c\frac{\prod_{k=1}^{m}(z - z_k)}{\prod_{k=1}^{n}(z - p_k)}$$

The filter amplification and phase shift can be characterized at each frequency using Bode diagram:

$$\left|G\left(e^{j\theta}\right)\right| = |c|\frac{\prod_{k=1}^{m}\left|e^{j\theta} - z_k\right|}{\prod_{k=1}^{n}\left|e^{j\theta} - p_k\right|}$$

When $e^{j\theta}$ is close to a pole, the magnitude of the response rises (*resonance*).

When $e^{j\theta}$ is close to a zero, the magnitude falls (a *null*).

$$\angle G\left(e^{j\theta}\right) = \angle(c) + \sum_{k=1}^{m} \angle\left(e^{j\theta} - z_k\right) - \sum_{k=1}^{n} \angle\left(e^{j\theta} - p_k\right)$$

5 Design of Filters

5.1 Ideal Filters

The ideal filter with normalized cutoff frequency $\theta_c = \omega_c T$:

$$H_d\left(e^{j\theta}\right) = \begin{cases} 1 & |\theta| \le \theta_c \\ 0 & |\theta| > \theta_c \end{cases}$$

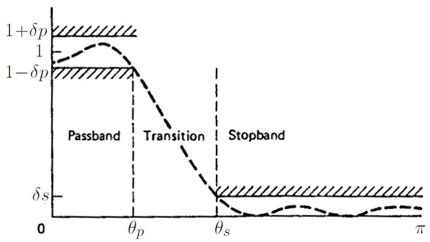
With sampling time T the max frequency is $\omega_{\max} = \frac{\pi}{T}$. Necessarily $\omega_c < \omega_{max}$.

Ideal filter can't be implemented due to non-causal impulse response $h_d(k) \neq 0$ for $k < 0$.

5.2 Realizable Filters

A typical filter specification must specify maximum permissible deviations from the ideal:

1. Band edge frequencies or corner frequencies θ_p and θ_s.
2. Maximum passband ripple $20\log_{10}(1+\delta p)$ dB.
3. Minimum stopband attenuation $-20\log_{10}(\delta s)$ dB.

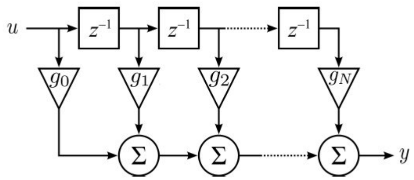


5.3 FIR Filters

FIR filter G can be derived by shift and truncation at $N+1$ samples of the ideal response $\{h_k\}$.

$$G(z) = \sum_{k=0}^{N} g_k z^{-k} \qquad g_k = h_{k-N/2}$$

FIR filters are simple, feedforward, inherently stable and can be realized efficiently in hardware (FFT). Causality is recovered. Truncation of small samples has modest impact.



Truncation is equivalent to multiplication by a window $g_k = h_k w_k$:

$$w_k = \begin{cases} 1 & 0 \le k \le N \\ 0 & \text{otherwise} \end{cases}$$

Using duality of multiplication and convolution:

$$G\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta$$

Transition band symmetric at cutoff ω_c. Transition bandwidth is related to mainlobe and reduces as $N \to \infty$. Ripples of G are related to the area under sidelobes, which remains constant as N increases.

The window method:

1. Select a suitable window function w_k.
2. Specify an ideal frequency response H.
3. Compute the coefficients of the ideal filter h_k.
4. Multiply the ideal coefficients by the window function to give the filter coefficients and delay to make causal.

The ideal bandpass filter:

$$H\left(e^{j\theta}\right) = \begin{cases} 1 & \omega_1 \le \theta \le \omega_2 \\ 0 & \text{otherwise} \end{cases}$$
$$= H_{d,\omega_2}\left(e^{j\theta}\right) - H_{d,\omega_1}\left(e^{j\theta}\right)$$

Linear phase $G\left(e^{j\theta}\right) = \left|G\left(e^{j\theta}\right)\right| e^{-j\theta\frac{N}{2}}$ is achieved if $\mathbf{g}_k = \mathbf{g}_{N-k}$. The window method gives *linear phase filters*. The coefficients of all window functions satisfies $w_k = W_{N-k}$.

Design by optimization: Given the ideal filter H and the weighting function W, find the optimal filter G of length N such that, given $E(\theta) = W(\theta)[H(\theta) - G(\theta)]$, G minimizes the least-squares error $\int_{-\pi}^{\pi} E^2(\theta)d\theta$ or the max error $\sup_{-\pi\le\theta\le\pi}|E(\theta)|$.

5.4 IIR Filters

IIR filters have finite polynomial representation but infinite impulse response:

$$A(z)Y(z) = B(z)U(z)$$

$$G(z) = \frac{B(z)}{A(z)} = \sum_{k=0}^{\infty} g_k z^{-k}$$

Discretization by response matching: Taking a continuous time filter with Laplace transfer function $G_c(s)$:

1. *Impulse invariance:*

$$G(z) = \mathcal{Z}\left(\mathcal{L}^{-1}\left(G_c(s)\right)_{t=kT}\right)$$

2. *Step response invariance:*

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G_c(s)}{s}\right)_{t=kT}\right)$$

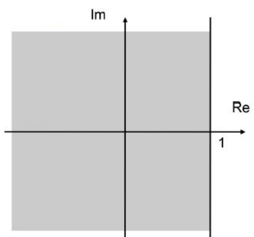
3. *Ramp invariance:*

$$G(z) = \frac{(z-1)^2}{Tz}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G_c(s)}{s^2}\right)_{t=kT}\right)$$

Discretization by algebraic transformations: $H(z) = H_c(s)_{s=\psi(z)}$ where $\psi(\cdot)$ is given by:

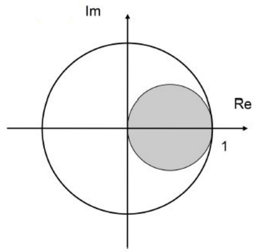
1. *Euler's method* or *Forward difference:*

$$s = \frac{z-1}{T}$$



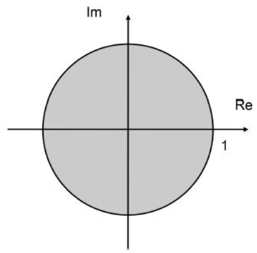
2. *Backward difference:*

$$s = \frac{z-1}{zT}$$



3. *Bilinear transformation* or *Tustin's transformation:*

$$s = \frac{2}{T}\frac{z-1}{z+1}$$



Backward difference and Tustin transformations applied to stable continuous systems result in stable discrete time systems (all the left plane poles get mapped into the unit disk). Not necessarily true for Euler's method.

Bilinear transform: Stability is preserved with frequency warping:

$$G\left(e^{j\theta}\right) = G_c(j\tan(\theta/2))$$

The frequency response of the analog filter at ω is mapped into the frequency response of the digital filter at $\theta = 2\arctan(\omega)$.

Transformation between different filter types: Assuming a lowpass prototype with cutoff at 1:

1. *Lowpass to Lowpass:* Set $s = \frac{\bar{s}}{\omega_c}$ to change the cutoff frequency to ω_c.
2. *Lowpass to Highpass:* Set $s = \frac{\omega_c}{\bar{s}}$ to get highpass with cutoff frequency at ω_c.
3. *Lowpass to Bandpass:* Set $s = \frac{\bar{s}^2+\omega_l\omega_u}{\bar{s}(\omega_u-\omega_l)}$ to get bandpass with lower cutoff at ω_l and upper cutoff at ω_u.
4. *Lowpass to Bandstop:* Set $s = \frac{\bar{s}(\omega_u-\omega_l)}{\bar{s}^2+\omega_l\omega_u}$ to get bandstop with lower cutoff at ω_l and upper cutoff at ω_u.

6 Design of Controllers

6.1 Control Design

Control in open loop:

$$W(z) = G(z)K(z)$$

Control design in closed loop:

$$W(z) = \frac{G(z)K(z)}{1+G(z)K(z)}$$

The Encirclement Property: Number of counterclockwise encirclements of the origin by $F\left(e^{j\theta}\right)$ as θ increases from 0 to 2π = number of zeros of $F(z)$ inside the unit circle − number of poles of $F(z)$ inside the unit circle.

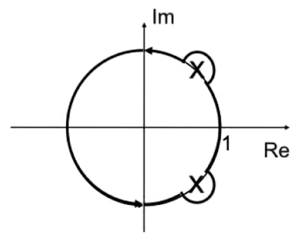
6.2 Closed Loop Stability

$$Y(z) = \frac{KG(z)}{1+KG(z)}R(z)$$

Number of counterclockwise encirclements of the origin by $1+KG\left(e^{j\theta}\right)$ = number of closed-loop poles inside the unit circle − number of open-loop poles inside the unit circle.

Nyquist stability criterion: The closed-loop system will be stable if (and only if) the number of counter clockwise encirclements of the $-1/K$ point by $G\left(e^{j\theta}\right)$ as θ increases from 0 to 2π = the number of open-loop unstable poles.

Closing the locus: It is customary to indent the path of z around the poles on the unit circle with a small semi-circular excursion outside the unit circle. The right turn in the z-plane then gives a right turn in the G-plane and a circular arc of large radius is produced which continues for $m\pi$ radians where m is the multiplicity of the pole on the unit circle.



Asymptotes: If there is an open-loop pole of multiplicity one on the unit circle then the Nyquist diagram will be asymptotic to a straight line as it tends to infinity.

7 Interfaces

7.1 Continuous and Discrete Interfaces

Analog-to-digital converter (ADC): Takes a continuous time signal $u(t)$, which is assumed to be continuous, and sample it to produce the number sequence $u(kT)$. T is the *sampling time* and ADC is also termed *sampler*.

Digital-to-analog converter (DAC): Take the number sequence $u(kT)$ and produces a continuous time signal $u(t)$.

Hybrid: $G(s)$ linear continuous system with discrete input and output before DAC and after ADC. The transfer function $G(z)$ from u to y is:

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kT\geq 0}\right)$$

7.2 Sampling

Sampling of signal with impulse response $g(t)$:

$$\begin{aligned} g_s(t) &= g(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \\ &= g(t)\frac{1}{T}\sum_{n=-\infty}^{\infty}e^{jn\omega_0 t} \quad \omega_0 = \frac{2\pi}{T} \end{aligned}$$

Periodicity of the sampled signal spectrum:

$$G_s(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty}G(j(\omega - n\omega_0))$$

Shannon theorem: A continuous-time signal $g(t)$ width bandwidth $\omega_{\max}$ can be reconstructed exactly from its sample version $g_s(t)$ if the sampling time satisfies $\omega_{\max} < \omega_0/2 = \pi/T$.

8 Discrete Fourier Transform

8.1 Transform Comparison

Z-transform:

$$X(z) = \mathcal{Z}(x) = \sum_{k=0}^{\infty}x_k z^{-k}$$

Discrete time Fourier transform (DTFT): Fourier transform of a sampled signal, sampling time T:

$$\overline{x}_\omega = \text{DTFT}(x) = \sum_{k=-\infty}^{\infty}x_k e^{-j\omega T k}$$

If $x_k = 0$ for $k<0$ then $\overline{x}_\omega = X(z)\,|_{z=e^{j\omega T}}$.

Discrete Fourier transform (DFT):

$$\overline{x}_p = \sum_{k=0}^{N-1}x_k e^{-j\frac{2\pi}{N}pk}$$

For truncated signals $x_k = 0$ for $k<0$ and $k\geq N$, $\overline{x}_p = X(z)\,|_{z=e^{-j\frac{2\pi}{N}p}}$.

Periodicity: $\overline{x}_p = \overline{x}_{p+N}$

$$\begin{aligned} \overline{x}_p &= \begin{bmatrix} e^{-j\frac{2\pi}{N}p\cdot 0} \\ e^{-j\frac{2\pi}{N}p\cdot 1} \\ \vdots \\ e^{-j\frac{2\pi}{N}p\cdot(N-1)} \end{bmatrix}^T \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \\ &= b(p,N)'x \end{aligned}$$

$\overline{x}_p$ is the projection of x on the base $b(p,N)$.

$$\begin{aligned} \overline{x} = \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \vdots \\ \overline{x}_{N-1} \end{bmatrix} &= \begin{bmatrix} b(0,N)' \\ b(1,N)' \\ \cdots \\ b(N-1,N)' \end{bmatrix} x \\ &= B(N)x \end{aligned}$$

Inverse DFT:

$$\begin{aligned} x_n &= \frac{1}{N}\sum_{p=0}^{N-1}\overline{x}_p e^{j\frac{2\pi}{N}pn} \\ &= \frac{1}{N}b(-n,N)'\overline{x} \end{aligned}$$

Using matrix product:

$$\begin{aligned} x &= \frac{1}{N}\begin{bmatrix} b(0,N)' \\ b(-1,N)' \\ b(-N+1,N)' \end{bmatrix}\overline{x} \\ &= \frac{1}{N}B(N)^* x = B(N)^{-1}\overline{x} \end{aligned}$$

Properties of the DFT:

1. *Periodicity:* $\overline{X}_p = \overline{X}_{p+N}$.
2. *Linearity:* If $z = (x+y)$ then $\overline{z} = \overline{x} + \overline{y}$.
3. *Symmetry:* If x is a real sequence then $\overline{x}_p = \overline{x}^*_{-p} = \overline{x}^*_{N-p}$.

8.2 Circular Convolution

For signal $\{x_k\}$ and FIR filter with impulse response $\{g_k\}$, the inverse DFT of the *product* of the DFTs is the circular convolution of x and g.

$$\overline{y}_p = \overline{g}_p\overline{x}_p, \quad \{\overline{y}_p\} \xrightarrow{iDFT} \{y_m\}$$

$$y_m = \sum_{k=0}^{N-1} g_k x_{\text{ mod }(m-k,N)}$$

mod $(k-n,N)$ denotes $k-n$ in modulo N arithmetic.

Filter response of FIR filter with $M+1 \ll N$ nonzero coefficients using standard linear convolution:

$$y_m = \sum_{k=0}^{\infty} g_k x_{m-k}$$

If $M \leq m < N$, then standard convolution = circular convolution:

$$\begin{aligned} \sum_{k=0}^{\infty} g_k x_{m-k} &= \sum_{k=0}^{M} g_k x_{m-k} \\ &= \sum_{k=0}^{N-1} g_k x_{mod(m-k,N)} \end{aligned}$$

8.3 Fast Fourier Transform

Fast Fourier Transform (FFT):

$$\overline{x} = \text{FFT}(x)$$

Inverse FFT:

$$x = \frac{1}{N}\text{FFT}(\overline{x}^*)^*$$

(The End)