Differentiation of Scalar Fields

Scalar Functions

We express that ϕ is a function of one independent variable x by writing $\phi = f(x)$ or $\phi = \phi(x)$.

If ϕ is a function of two independent variables, x and y, we write $\phi = \phi(x, y)$.

Differentiation of a Scalar Function

If ϕ is a function of one variable, then we have the following definition of the derivative:

$\frac{d\phi}{dx} = \lim_{\delta x \to 0} \left(\frac{\phi(x+\delta x) - \phi(x)}{\delta x} \right)$

As $\delta x \to 0$, the derivative becomes tangent to the curve of $\phi(x)$.

If ϕ is a function of two variables, we now have **partial derivatives** defined as:

$\frac{\partial\phi}{\partial x} = \lim_{\delta x \to 0} \left(\frac{\phi(x+\delta x, y) - \phi(x, y)}{\delta x} \right)$

$\frac{\partial\phi}{\partial y} = \lim_{\delta y \to 0} \left(\frac{\phi(x, y+\delta y) - \phi(x, y)}{\delta y} \right)$

$\frac{\partial\phi}{\partial x}$ is the rate of change of ϕ with x when y is held constant. It is the slope of the curve formed by slicing the surface $\phi = \phi(x, y)$ along a plane at constant y.

Total Differentials

The **total differential** of a scalar function is defined as:

$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy$

The substantive (material) derivative:

The acceleration (vector) of a fluid particle as it moves through a velocity field can be calculated as a total derivative.

$\frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial t} + V_x \frac{\partial\mathbf{V}}{\partial x} + V_y \frac{\partial\mathbf{V}}{\partial y}$

Chain Rule for Scalar Functions

If ϕ is a function of two variables, $\phi = \phi(x, y)$, we can find $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ where $x = x(u, v)$ and $y = y(u, v)$.

$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u}$

$\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v}$

Taylor Series for Scalar Functions

For a function of two independent variables, $\phi = \phi(x, y)$, we can use a Taylor expansion to find $\phi(x, y)$.

$\phi(x, y) = \phi(x_0, y_0) + (x - x_0)\left(\frac{\partial\phi}{\partial x}\right)_0 + (y - y_0)\left(\frac{\partial\phi}{\partial y}\right)_0 + \frac{(x-x_0)^2}{2!}\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 + (x - x_0)(y - y_0)\left(\frac{\partial^2\phi}{\partial x \partial y}\right)_0 + \frac{(y-y_0)^2}{2!}\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 + \cdots$

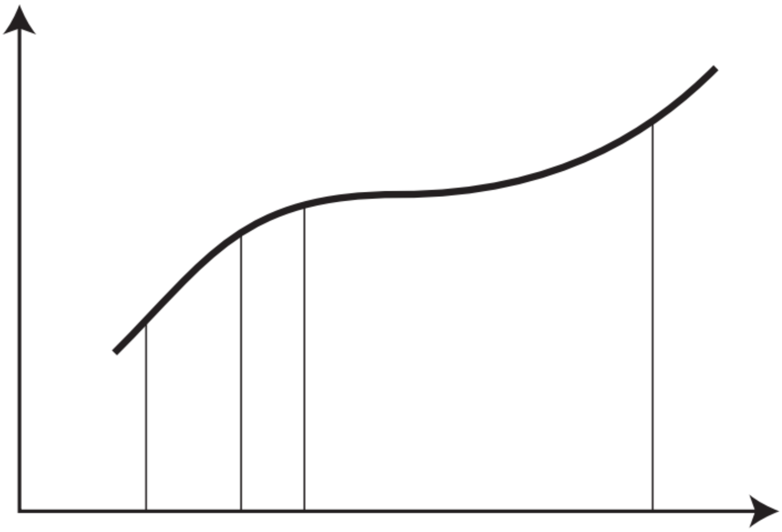
Integration of a Scalar Field

Integration of a Scalar Function

If ϕ is a scalar function of one independent variable, we define the integration of ϕ between the limits of $x = a$ and $x = b$ as:

$\int_a^b \phi dx = \lim_{\delta x_i \to 0} \sum_{i=1}^{\infty} \phi_i \delta x_i$

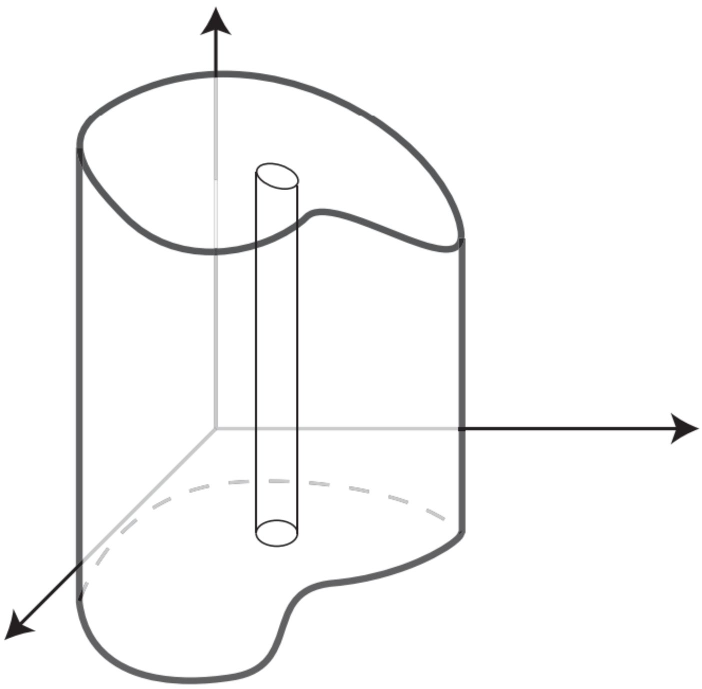
The result of this integration is the “area under the curve”.



If ϕ is a function of two independent variables, we define the integration as:

$\int_A \phi dA = \lim_{\delta A_i \to 0} \sum_{i=1}^{\infty} \phi_i \delta A_i = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi dx dy$

The result of this integration is the volume enclosed by the surface $\phi = \phi(x, y)$, the area A (on the x–y plane), and the 'vertical curtain' connecting the boundary of A with the ϕ surface.



Change of Variable and the Jacobian

If ϕ is a function of two independent variables, $\phi = \phi(x, y)$, and we wish to evaluate the following integral, it may be easier to switch to new independent variables u and v such that $x = x(u, v)$ and $y = y(u, v)$.

$I = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \phi(x, y) dx dy = \int_{v_1}^{v_2} \int_{u_1}^{u_2} \phi(u, v) \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right| du dv$

The expression inside the $|\cdots|$ is called the **Jacobian**.

$J = \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)} \right)^{-1}$

The Gradient of a Scalar Field

The Vector Operator 'Del'

In Cartesian coordinates, the **vector operator**, ∇, is defined by:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

If ϕ is a scalar function $\phi = \phi(x, y, z)$, the gradient of ϕ (or 'grad ϕ') is a **vector** and can be defined as:

$$\nabla_\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

If f and g are both scalar fields:

$$\nabla(f + g) = \nabla f + \nabla g$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

Physical Interpretation of the Gradient

The vector gradient is the 3-D equivalent of the slope of a curve in 1-D.

The directional derivative:

$$\frac{d\phi}{ds} = \nabla_\phi \cdot \hat{\mathbf{n}}$$

> If $\hat{\mathbf{n}}$ lies on the surface of constant ϕ, $\frac{d\phi}{ds} = \nabla_\phi \cdot \hat{\mathbf{n}} = 0$.

> The magnitude of $\frac{d\phi}{ds}$ is greatest when $\hat{\mathbf{n}}$ is parallel to ∇_ϕ.

> The direction of ∇_ϕ is always in the direction of increasing ϕ ("∇_ϕ always points up hill").

The Scalar Operator and Substantive Derivative

$\mathbf{V} \cdot \nabla$ is a **scalar operator** that can act on either a scalar field or a vector field.

$$\mathbf{V} \cdot \nabla = V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z}$$

The rate of change of temperature, as measured by a probe moving through a time-varying temperature field $T = T(x, y, z, t)$, is given by the total derivative:

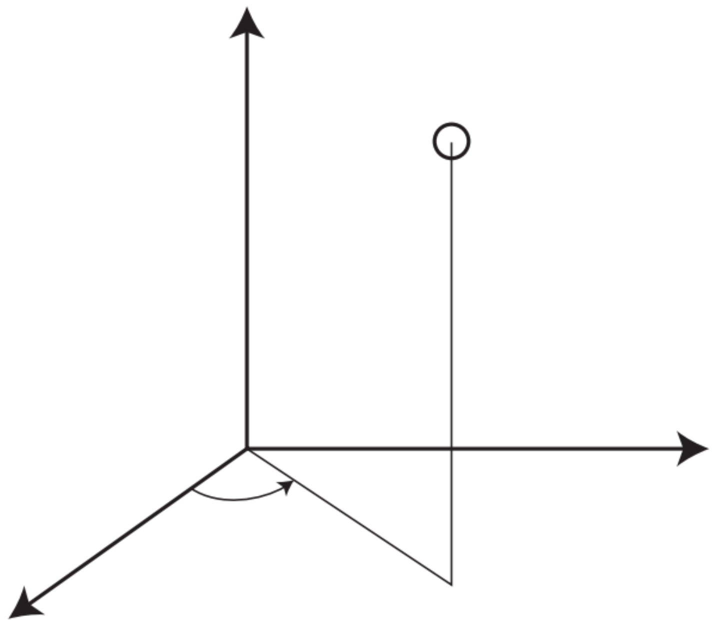
$$\frac{dT}{dt} = (V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} + V_z\frac{\partial T}{\partial z}) + \frac{\partial T}{\partial t} = (\mathbf{V} \cdot \nabla)T + \frac{\partial T}{\partial t}$$

Gradient in Non-Cartesian Coordinate Systems

∇ in other coordinate systems can be defined by $\delta f = \nabla f \cdot \delta\mathbf{r}$.

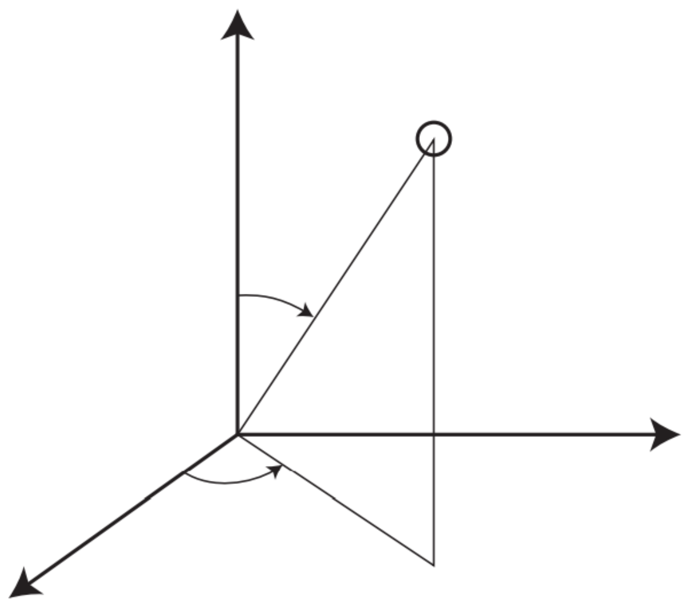
For **cylindrical polar coordinates** (r, θ, z):

$$\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}$$



For **spherical polar coordinates** (r, θ, ϕ):

$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi}$



The Divergence of a Vector Field

Definition and Identities

The divergence of a vector field $\mathbf{V}$ is obtained by taking the dot product of the vector operator ∇ with $\mathbf{V}$ and is written $\nabla \cdot \mathbf{V}$ (or 'div $\mathbf{V}$'). The divergence of a vector field is a **scalar**.

In Cartesian coordinates:

$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Physical Interpretation of the Divergence

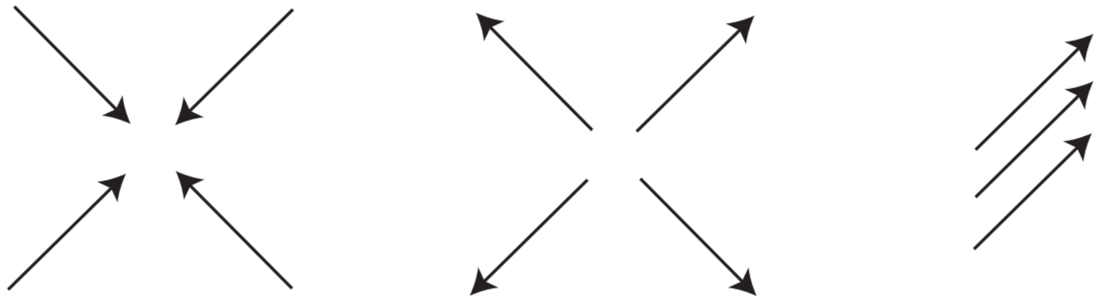
The divergence is an important quantity that is linked to the conservation of physical properties such as mass, momentum, energy, magnetic flux, etc.

The conservation of mass in vector form, valid for any coordinate system:

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})$

Solenoidal Vector Fields

A vector field where the divergence is everywhere zero is called a **solenoidal** field. In a solenoidal field, the net efflux of the vector field from a volume element δv is zero. The flux entering the volume element is the same as flux leaving the element: there are no 'sources' or 'sinks' of the vector field within the element.



Divergence in Non-Cartesian Coordinate Systems

For **cylindrical polar coordinates** (r, θ, z):

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$$

For **spherical polar coordinates** (r, θ, ϕ):

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial (r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (V_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}$$

Divergence and Gradient Combined

The operator $\nabla \cdot \nabla$ is a second order scalar differential operator and is usual written ∇^2('del squared') and is known as the Laplacian (the equation $\nabla^2 \phi$ is called Laplace's equation).

In Cartesian coordinates, the Laplacian is defined as:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

When the vector field is governed by a scalar potential, and is also solenoidal, it follows that:

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$