Energy, Power and Delta Functions

Power and Energy

The total **energy** content of a signal $f\left(t\right)$ is defined as:

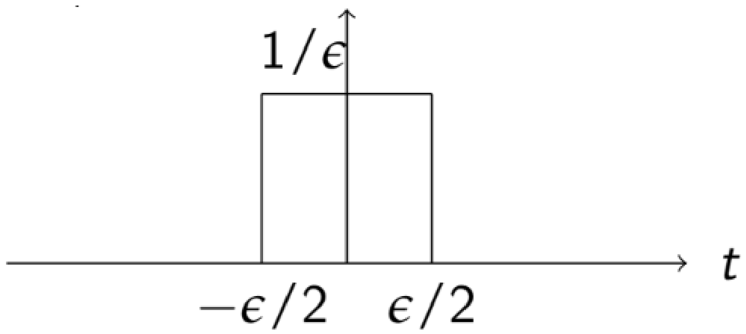
$E_f=\lim_{T\to\infty}\int_{-T/2}^{T/2}\left|f\left(t\right)\right|^2\mathrm{dt}=\int_{-\infty}^{\infty}\left|f\left(t\right)\right|^2\mathrm{dt}$

The average **power** content of the signal is defined as:

$P_f=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\left|f\left(t\right)\right|^2\mathrm{dt}$

Delta Functions

The δ-function, $\delta\left(t\right)$ can be defined as a limiting case where f_1 is a rectangular pulse:



$\delta\left(t\right)=\lim_{\epsilon\to 0}f_1\left(t;\epsilon\right)$

The delta-function can also be defined as limiting cases of other functions:

$f_2\left(t;\epsilon\right)=\frac{\epsilon}{\epsilon^2\pi^2+t^2}$

$f_3\left(t;a\right)=\frac{\sin(\mathrm{at})}{\pi\mathrm{t}}$

$\delta\left(t\right)=\lim_{\epsilon\to 0}f_2\left(t;\epsilon\right)=\lim_{a\to\infty}f_3\left(t;a\right)$

Properties of the δ-function:

$\delta\left(t\right)=0\ for\ t\neq 0$

$\int_{-\infty}^{\infty}\delta\left(t\right)\mathrm{dt}=1$

The Sifting Property

$\int_{-\infty}^{\infty}g\left(t\right)\delta\left(t-a\right)\mathrm{dt}=g\left(a\right)$

A Useful Integral

$\int_{-\infty}^{\infty}\exp(\mathrm{j}\omega\mathrm{t})\mathrm{d}\omega=\lim_{A\to\infty}\int_{-A}^{A}\exp(\mathrm{j}\omega\mathrm{t})\mathrm{d}\omega=\lim_{A\to\infty}\left[\frac{e^{\mathrm{j}\omega t}}{\mathrm{jt}}\right]_{-A}^{A}=\lim_{A\to\infty}\left(2\frac{\sin(\mathrm{At})}{t}\right)=2\pi\delta\left(t\right)$

Fourier Series

Fourier Series

Any function, $g(t)$, which is **periodic** in the interval $[-\pi, \pi]$ has a real Fourier series representation given by:

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(\mathrm{nt}) + b_n \sin(\mathrm{nt})\}$$

The coefficients are given by:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(\mathrm{nt})\, \mathrm{dt}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(\mathrm{nt})\, \mathrm{dt}$$

Any periodic function can be formed from a linear combination of the functions $\cos(\mathrm{nt})$ and $\sin(\mathrm{nt})$.

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{jnt}}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, e^{-jnt} \mathrm{dt}$$

Relationship between real and complex coefficients:

$$2c_n = a_n - jb_n$$

$$2c_{-n} = 2c_n^* = a_n + jb_n$$

$$a_n = c_n^* + c_n$$

$$jb_n = c_n^* - c_n$$

$$2|c_n| = \sqrt{a_n^2 + b_n^2}$$

When a periodic signal has period T rather than 2π and $\omega_0 = 2\pi/T$:

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{jn}\omega_0 t}$$

$$c_n = \frac{1}{T} \int_{\alpha}^{\alpha+T} g(t)\, e^{-jn\omega_0 t} \mathrm{dt}$$

Properties of Fourier Series

Modify $g(t)$ by scaling the amplitude by a factor of a, shifting it along the axis by b and changing the period to $T' = \beta T$:

$$g'(t) = ag\left(\frac{t-b}{\beta}\right) = \sum_{n=-\infty}^{\infty} \left\{ac_n e^{-\frac{j\omega_0 \mathrm{nb}}{\beta}}\right\} e^{\frac{j\omega_0 \mathrm{nt}}{\beta}} = \sum_{n=-\infty}^{\infty} c_n' e^{j\omega_0' \mathrm{nt}}$$

$$c_n' = ac_n e^{-\frac{j\omega_0 \mathrm{nb}}{\beta}}$$

$$\omega_0' = \frac{\omega_0}{\beta}$$

Interpretation of Fourier Coefficients

The component with frequency ω_0 is known as the **fundamental frequency**, or **first harmonic**, $n\omega_0$ is the n-th harmonic.

The amplitudes of the harmonics are defined by:

$$r_n = \sqrt{a_n^2 + b_n^2} = 2|c_n| \equiv |c_{-n}| + |c_n|$$

$$r_0 = \frac{a_0}{2} = c_0$$

Parseval's Theorem for Fourier series:

$\frac{1}{T}\int_0^T |g(t)|^2 \mathrm{dt} = \sum_{-\infty}^{\infty} |c_n|^2$

Fourier Transforms

Fourier Transforms

$F(\omega) = \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t)\, e^{-j\omega t} \mathrm{dt} = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} \mathrm{dt}$

The Fourier transform is often referred to as the **Fourier spectrum** of a signal.

The Inverse Fourier Transform (IFT)

$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{\mathrm{j}\omega t} \mathrm{d}\omega$

Properties of the Fourier transform

The Fourier transform is linear:

$af_1(t) + bf_2(t) \overset{\mathrm{FT}}{\leftrightarrow} aF_1(\omega) + bF_2(\omega)$

Time Scaling (Similarity) Theorem:

If we stretch in the time domain, we contract in the frequency domain.

$f(\alpha \mathrm{t}) \overset{\mathrm{FT}}{\leftrightarrow} \frac{1}{\alpha} F(\omega)$

Heisenberg-Gabor Principle:

If any function $f(t)$ has time duration T, and its Fourier transform $F(\omega)$ has frequency bandwidth B, then the Time-Bandwidth product $TB \geq 1$.

Modulation Theorem:

$f(t - t_0) \overset{\mathrm{FT}}{\leftrightarrow} F(\omega)\, e^{-j\omega t_0}$

$e^{-j\omega t_0} f(t) \overset{\mathrm{FT}}{\leftrightarrow} F(\omega - \omega_0)$

$f(t)\cos(\omega_0 t) \overset{\mathrm{FT}}{\leftrightarrow} \frac{1}{2}F(\omega - \omega_0) + \frac{1}{2}F(\omega + \omega_0)$

Differentiate with respect to time:

$f^n(t) \overset{\mathrm{FT}}{\leftrightarrow} (\mathrm{j}\omega)^n F(\omega)$

The **dual** Fourier transform pair:

$f(t) \overset{\mathrm{FT}}{\leftrightarrow} g(\omega)$

$g(t) \overset{\mathrm{FT}}{\leftrightarrow} 2\pi f(-\omega)$

The Multiplication Theorem:

$\int_{-\infty}^{\infty} f_1(t)\, f_2^*(t)\, \mathrm{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\, F_2^*(\omega)\, \mathrm{d}\omega$

Parseval's Theorem:

$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 \mathrm{d}\omega$

The amount of energy in a system can be found by integrating in the time domain or in the spectral domain.

The **Convolution Theorem** for Fourier transforms:

$f(t) * g(t) \overset{\text{FT}}{\leftrightarrow} F(\omega)\, G(\omega)$

$2\pi f(t)\, g(t) \overset{\text{FT}}{\leftrightarrow} F(\omega) * G(\omega)$

Laplace Transform

$\overline{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st} \mathrm{dt}$

For a causal linear time-invariant system, $h(t) = 0$ for any $t < 0$.

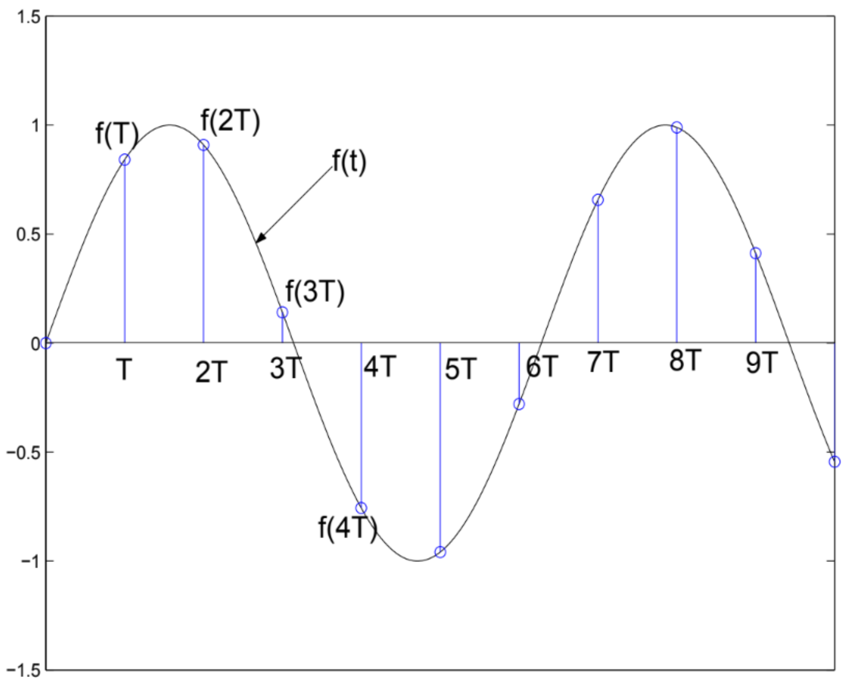
$h(t) \overset{\text{LT}}{\leftrightarrow} \overline{h}(s)$

$h(t) \overset{\text{FT}}{\leftrightarrow} F(\omega)$

$H(\omega) = \overline{h}(\mathrm{j}\omega) = Frequency\ response$

Sampling Theory

Sampling and Aliasing

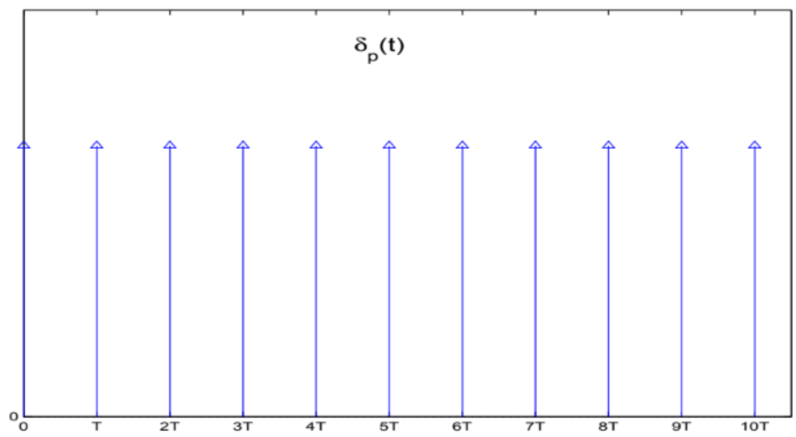


In Digital sampling, for a continuous time signal, choose a sampling interval T and read off the value of $f(t)$ at times nT. The obtained values $f(\mathrm{nT})$ are the sampled version of $f(t)$.

The Sampling Theorem

$f_s(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - nT) = f(t)\, \delta_p(t)$

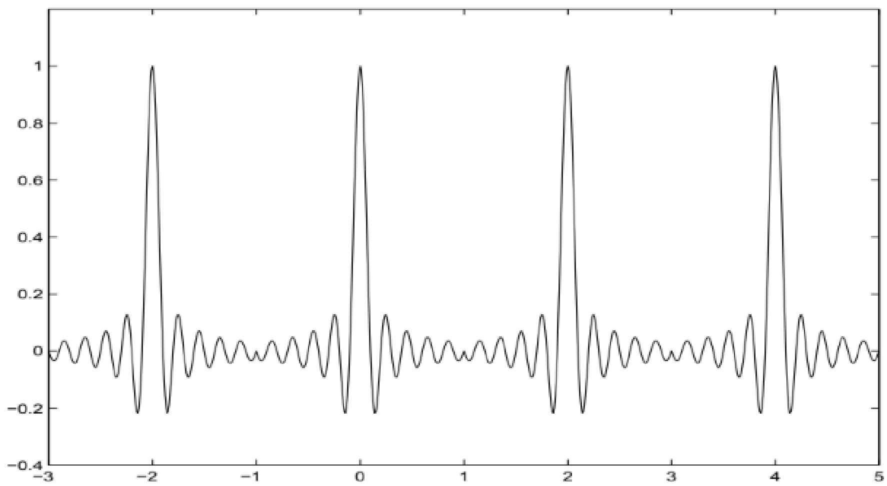
$\delta_p(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$



From the frequency shift theorem:

$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$

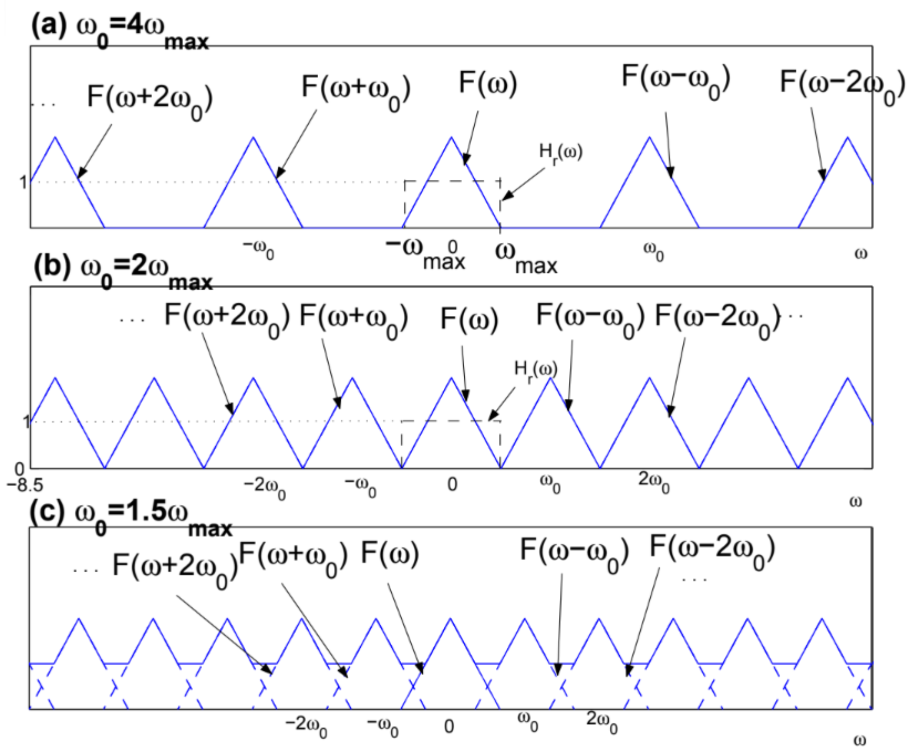
The Fourier transform of the sampled signal is $1/T$ times the Fourier transform of the continuous signal repeated every integer multiple of the sampling frequency and summed together.



Discrete-time Fourier Transform (DTFT)

$F_s(\omega) = \int_{-\infty}^{\infty} f_s(t)\, e^{-j\omega t} \mathrm{dt} = \sum_{n=-\infty}^{\infty} f(\mathrm{nT})\, e^{-jn\omega T}$

Nyquist Frequency and Reconstruction



The Nyquist Sampling Theorem:

> If a signal $f(t)$ has a maximum frequency content (or bandwidth) $\omega_{\max}$, then it is possible to reconstruct $f(t)$ perfectly from its sampled version of $f_s(t)$ provided the sampling frequency is at least $\omega_0 = 2\omega_{\max}$, the **Nyquist frequency**.

The repetitions of $F(\omega)$ in the sampled spectrum are known as **aliasing**.

When a signal is sampled at a rate less than ω_{Nyq}, the distortion due to the overlapping spectra is call **aliasing distortion**.

Ideal Reconstruction Filter

The ideal filter frequency response for perfect reconstruction is the rectangle pulse function.

$$H_r(\omega) = \begin{cases} T, & -\omega_{\max} < \omega < +\omega_{\max} \\ 0, & otherwise \end{cases}$$

For sampling at Nyquist frequency:

$$h_r(t) = \frac{\omega_{\max} T}{\pi} \text{sinc}(\omega_{\max} t) = \text{sinc}\left(\frac{\omega_0 t}{2}\right)$$

Multiplication in the frequency domain implies convolution in the time domain:

$$f(t) = \int_{-\infty}^{\infty} f_s(\tau)\, \text{sinc}\left(\frac{\omega_0 (t-\tau)}{2}\right) \text{d}\tau = \sum_{n=-\infty}^{\infty} f(\text{nT})\, \text{sinc}\left(\frac{\pi}{T}(t - nT)\right)$$

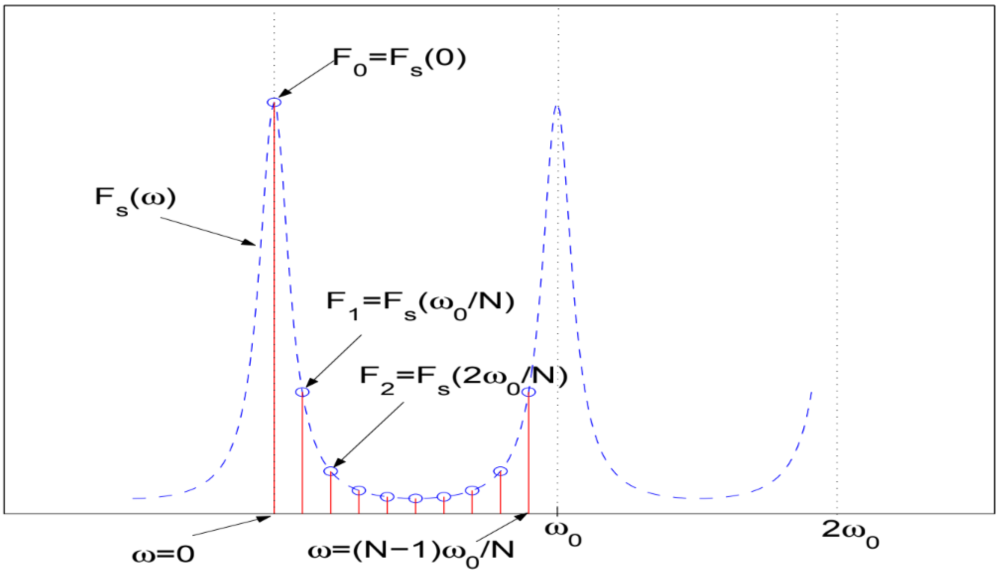
The Discrete Fourier Transform

Discrete Fourier Transform (DFT)

Consider only date points which lie within a finite range $[0,\ (N-1)T]$ and calculate only over a finite grid of frequencies $[0,\ (N-1)\omega_0/N]$.

$$f_n = f(\text{nT})$$

$$F_m = F_s\left(\frac{m\omega_0}{N}\right) = \sum_{n=0}^{N-1} f_n e^{-jnm2\pi/N}$$



The DFT is **periodic**, $F_{k+N} = F_k$, and for real signals, $F_{-k} = F_k$.

Inverse Discrete Fourier Transform

$f_n = \frac{1}{N} \sum_{n=0}^{N-1} F_m e^{-jmn2\pi/N}$