Impulse Responses, Step Responses and Transfer Functions

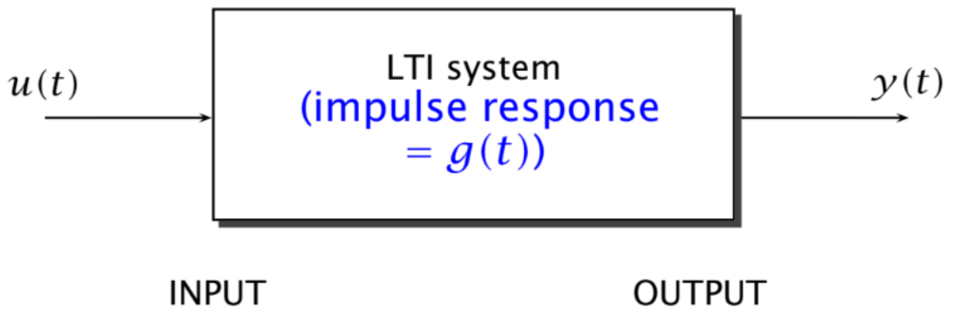
The Impulse and Step Responses

The impulse response of a system is the output of the system when the input is an impulse, $\delta(t)$, and all initial conditions are zero.

The step response of a system is the output of the system when the input is a step, $H(t)$, and all initial conditions are zero.

$$H\left(t\right)=\begin{cases}1\,if\,t>0\\0\,if\,t>0\end{cases}$$

The Convolution Integral



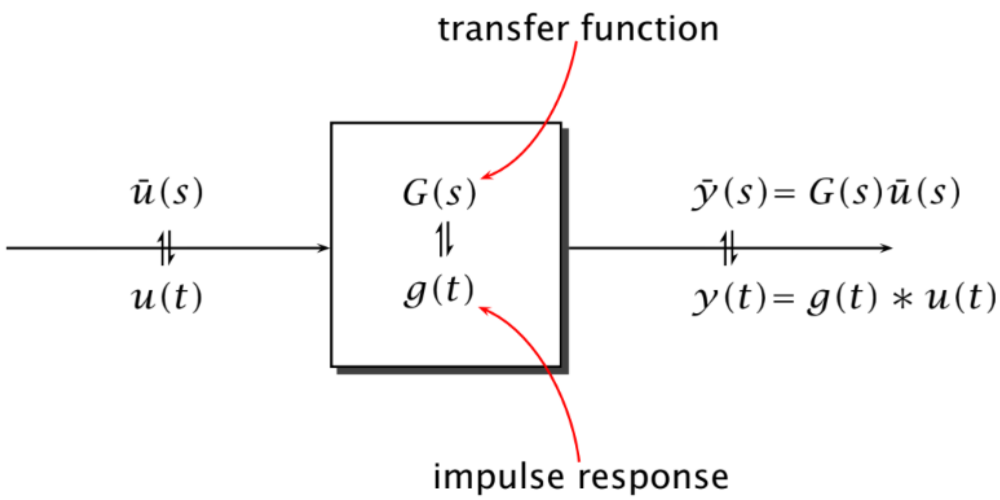
The response to the input $u(t)$ is given by:

$$y\left(t\right)=\int_{-\infty}^{\infty}u\left(\tau\right)g\left(t-\tau\right)\mathrm{d}\tau=u\left(t\right)*g\left(t\right)$$

If, in addition, $g\left(t\right)=0$ for $t<0$ (causality) and $u\left(t\right)=0$ for $t<0$ (standing assumption):

$$y\left(t\right)=\int_{0}^{t}u\left(\tau\right)g\left(t-\tau\right)\mathrm{d}\tau$$

The Transfer Function



If a linear system has input u and output y satisfying the ODE and if all initial conditions are zero:

$$\frac{d^2y}{dt^2}+\alpha\frac{\mathrm{dy}}{\mathrm{dt}}+\beta y=a\frac{\mathrm{du}}{\mathrm{dt}}+bu$$

$\overline{y}(s) = \frac{as+b}{s^2+\alpha s+\beta}\overline{u}(s) = G(s)\,u(s)$

Stability and Pole Locations

Asymptotic Stability

An LTI system is **asymptotically stable** if its impulse response $g(t)$ satisfies the condition:

$\int_0^\infty |g(t)|\,\mathrm{dt} < \infty$

Poles and the Impulse Response

For a general LTI system described by an ODE, is has a **rational** transfer function $G(s)$:

$G(s) = \frac{n(s)}{d(s)} = \frac{n(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$

For a **proper** $G(s)$:

$\deg[n(s)] \le \deg[d(s)]$

$G(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \ldots + \frac{\alpha_n}{s-p_n} + C$

The **residue** at $s = p_i$ is $\alpha_i = \lim_{s\to p_i}(s-p_i)\,G(s)$, assuming no repeated poles.

$g(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \ldots + \alpha_n e^{p_n t} + C\delta(t)$

If p is real, then e^{pt} is a real exponential with time constant $|1/p|$.

If p is complex, then $\Re(\alpha e^{\mathrm{pt}})$ gives either a damped or a growing sinusoid with time constant $|1/\sigma|$ and frequency ω. The imaginary part will cancel out with the contribution from p^*.

$\Re(\alpha e^{\mathrm{pt}}) = \Re\left(Ae^{\mathrm{j}\phi}e^{\mathrm{pt}}\right) = \Re\left(Ae^{\sigma t}e^{j(\omega t+\phi)}\right) = Ae^{\sigma t}\cos(\omega t+\phi)$

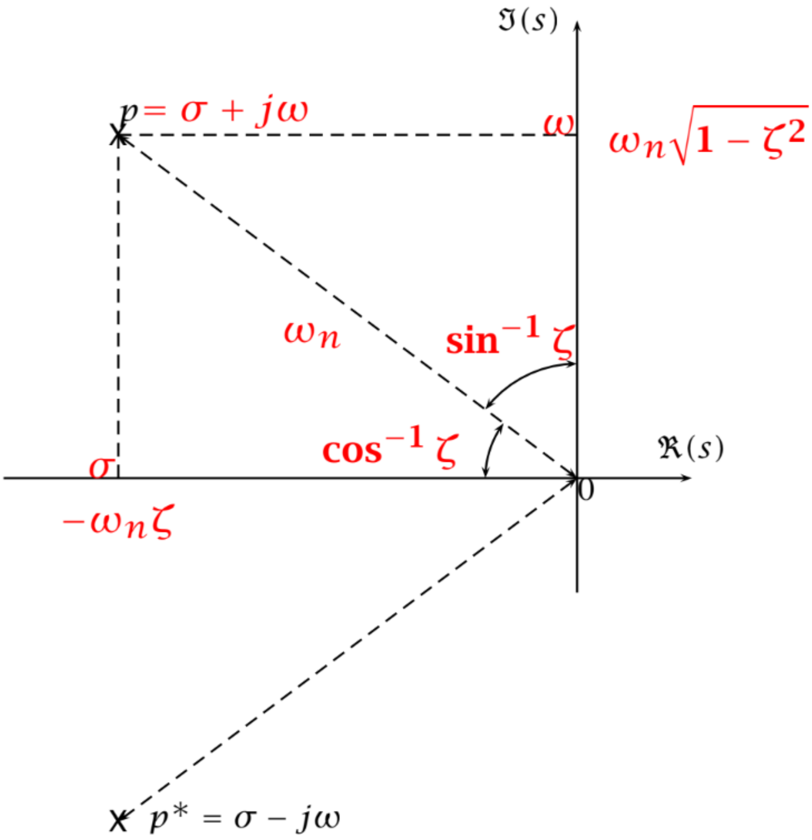
$\sigma = \Re(p)$

$\omega = \Im(p)$

Compare this with the impulse response of a second order system $Ce^{-\omega_n\zeta t}\sin(\omega_d t)$:

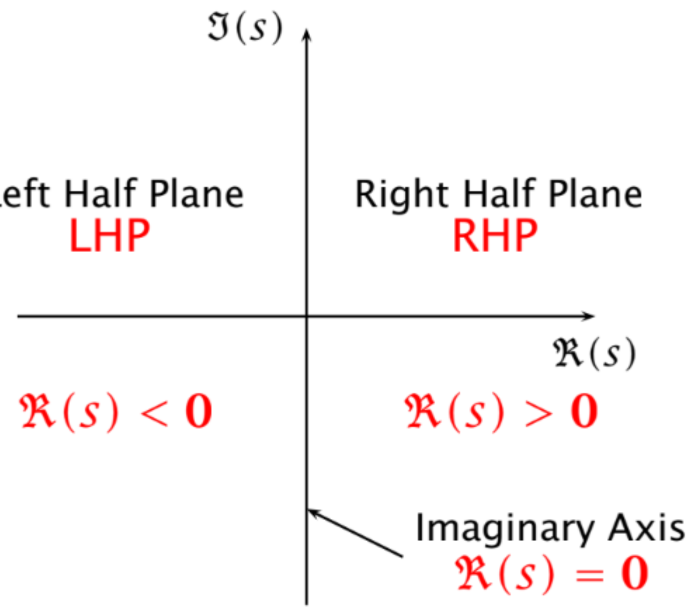
$\sigma = -\omega_n\zeta$

$\omega = \omega_d = \omega_n\sqrt{1-\zeta^2}$



Asymptotic stability and pole locations:

An LTI system with rational transfer function $G(s)$ is asymptotically stable if and only if all poles of $G(s)$ lie in the LHP.



Marginal Stability

An LTI system is marginally stable if it is not asymptotically stable, but there nevertheless exist numbers $A, B < \infty$ such that for all T:

$\int_0^T |g(t)|\,\mathrm{dt} < A + BT$

Instability

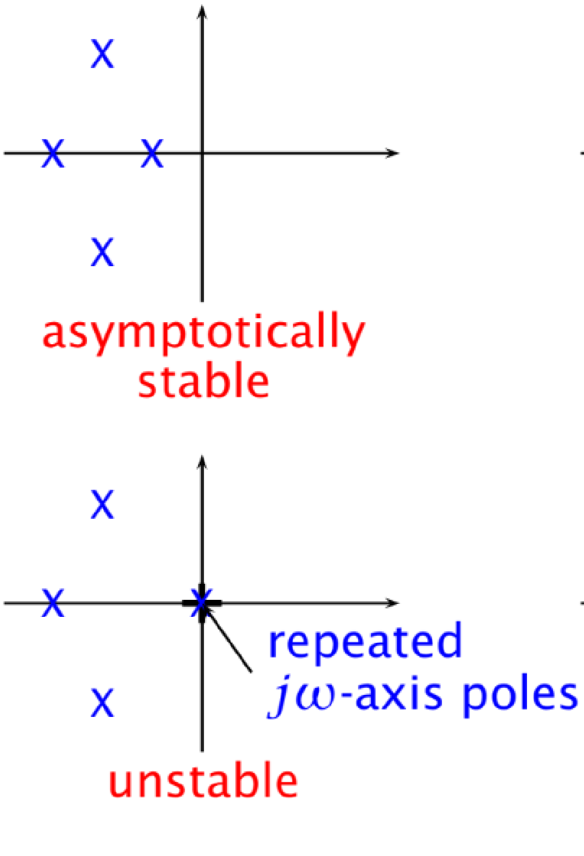
A system is unstable if it is neither asymptotically stable nor marginally stable.

Stability Theorem:

> A system is asymptotically stable if **all** its poles have negative real parts.

> A system is unstable if **any** pole has a positive real part, **or** if there are repeated poles on the imaginary axis.

> A system is marginally stable if it has one or more distinct poles on the imaginary axis, **and** any remaining poles have negative real parts.



Poles and the Transient Response

The **transient response** is the initial part of the (time domain) response of a system to a general input.

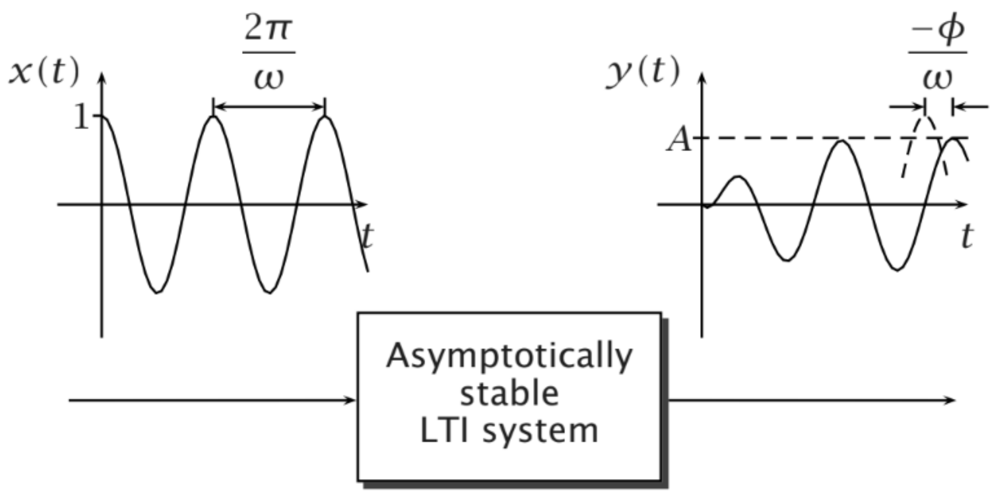
$$\overline{y}(s) = G(s)\overline{u}(s) = \frac{n(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)}\overline{u}(s) = \frac{\gamma_1}{s-p_1} + \frac{\gamma_2}{s-p_2} + \ldots + \frac{\gamma_n}{s-p_n} + extra\ terms$$

$$y(t) = \gamma_1 e^{p_1 t} + \gamma_2 e^{p_2 t} + \ldots + \gamma_n e^{p_n t} + extra\ terms$$

The response contains the same terms as the impulse response (although with different amplitudes) plus some extra terms due to particular characteristics of the input.

The Frequency Response

The Frequency Response



If the input to an asymptotically stable LTI system is a pure sinusoid then the steady state output will also be a pure sinusoid of the same frequency as the input but at a different amplitude and phase.

If system has the transfer function $G\left(s\right)$:

$A = \left|G\left(\mathrm{j}\omega\right)\right|$

$\phi = \arg G\left(\mathrm{j}\omega\right)$

Gain and Phase Shift

For an asymptotically stable system with input $u\left(t\right) = e^{\mathrm{j}\omega \mathrm{t}}$:

$\overline{y}\left(s\right) = G\left(s\right)\overline{u}\left(s\right) = \frac{n(s)}{d(s)}\frac{1}{s-j\omega} = \frac{\lambda_1}{s-p_1} + \frac{\lambda_2}{s-p_2} + \ldots + \frac{\lambda_n}{s-p_n} + \frac{\lambda_0}{s-j\omega}$

$y\left(t\right) = \lambda_1 e^{p_1 t} + \lambda_2 e^{p_2 t} + \ldots + \lambda_n e^{p_n t} + G\left(\mathrm{j}\omega\right)e^{\mathrm{j}\omega \mathrm{t}}$

Since all the poles of $G\left(s\right)$ have a negative real part, as $t \rightarrow \infty$:

$y_{\mathrm{tr}}\left(t\right) = \lambda_1 e^{p_1 t} + \lambda_2 e^{p_2 t} + \ldots + \lambda_n e^{p_n t} \rightarrow 0$

$\Re\left(y_{\mathrm{ss}}\left(t\right)\right) = \Re\left(G\left(\mathrm{j}\omega\right)e^{\mathrm{j}\omega \mathrm{t}}\right) = \Re\left(\left|G\left(\mathrm{j}\omega\right)\right|e^{j(\omega t + \arg G(\mathrm{j}\omega))}\right) = \left|G\left(\mathrm{j}\omega\right)\right|\cos(\omega t + \arg G\left(\mathrm{j}\omega\right))$

Frequency Response of an Arbitrary Linear System

The response of an asymptotically stable system to a sinusoidal input $x\left(t\right) = \Re\left(e^{\mathrm{j}\omega \mathrm{t}}\right) = \cos\omega \mathrm{t}$:

$y\left(t\right) = \Re\left(\int_0^t e^{\mathrm{j}\omega(t-\tau)}g\left(\tau\right)\mathrm{d}\tau\right) = \Re\left(\int_0^\infty e^{\mathrm{j}\omega(t-\tau)}g\left(\tau\right)\mathrm{d}\tau\right) - \Re\left(\int_t^\infty e^{\mathrm{j}\omega(t-\tau)}g\left(\tau\right)\mathrm{d}\tau\right)$

Second term is the **transient** part of the response, $y_{\mathrm{tr}}\left(t\right)$, as it decays to zero eventually and is transitory in nature.

$\left|y_{\mathrm{tr}}\left(t\right)\right| \leq \int_t^\infty \left|e^{\mathrm{j}\omega(t-\tau)}\right|\left|g\left(\tau\right)\right|\mathrm{d}\tau = \int_t^\infty \left|g\left(\tau\right)\right|\mathrm{d}\tau < \infty$

$\lim_{t\rightarrow\infty}\left|y_{\mathrm{tr}}\left(t\right)\right| = 0$

First term is the **steady-state** part of the response, $y_{\mathrm{ss}}\left(t\right)$, as it is part that remains when the transients have decayed.

$y_{\mathrm{ss}}\left(t\right) = \Re\left(e^{\mathrm{j}\omega \mathrm{t}}\int_0^\infty e^{-j\omega\tau}g\left(\tau\right)\mathrm{d}\tau\right) = \Re\left(e^{\mathrm{j}\omega \mathrm{t}}G\left(\mathrm{j}\omega\right)\right) = \left|G\left(\mathrm{j}\omega\right)\right|\cos(\omega t + \arg G\left(\mathrm{j}\omega\right))$

Plotting the Frequency Response

The Bode Diagram:

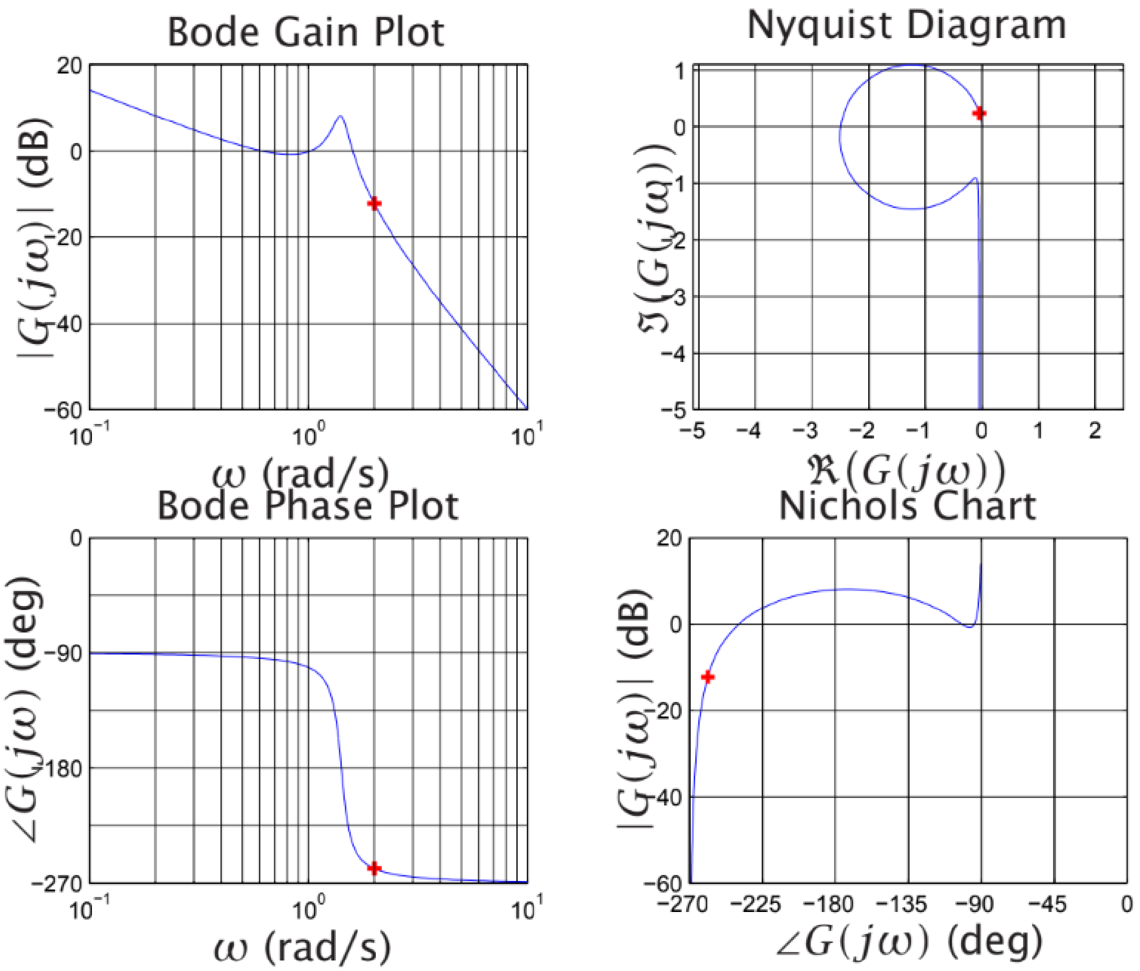
Two separate graphs, one of $|G(\mathrm{j}\omega)|$ against ω (on log-log axes), and one of $\angle G(\mathrm{j}\omega)$ (linear axis) against ω (log axis).

The Nyquist Diagram:

One single parametric plot, of $\Re(G(\mathrm{j}\omega))$ against $\Im(G(\mathrm{j}\omega))$ (on linear axes) as ω varies.

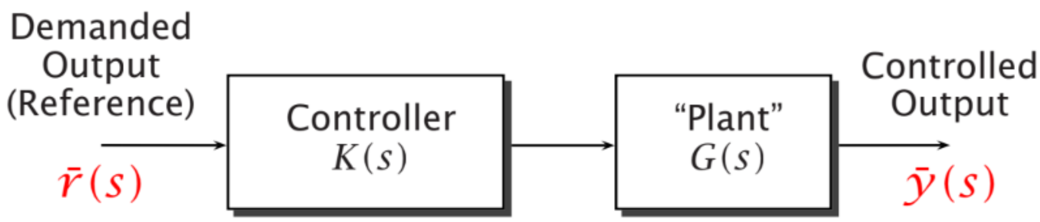
The Nichols Diagram:

One single parametric plot, of $|G(\mathrm{j}\omega)|$ (log axis) against $\angle G(\mathrm{j}\omega)$ (linear axis) as ω varies.



Feedback Control Systems

Open-Loop Control

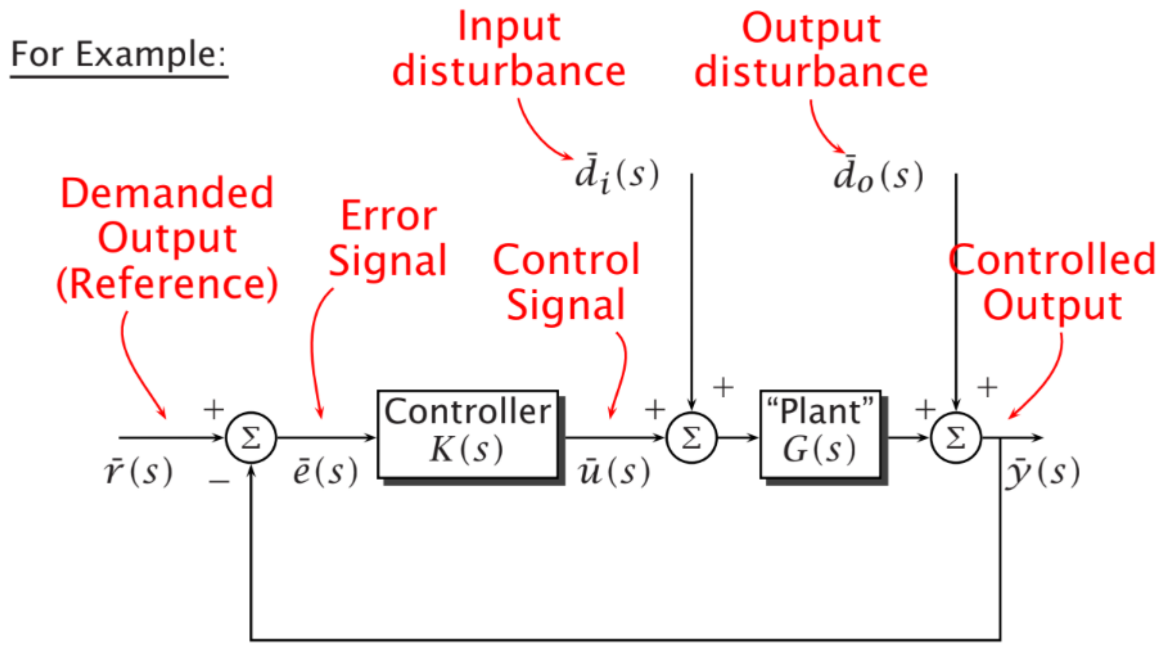


For a system with demanded output $\overline{r}(s)$ and controlled output $\overline{y}(s)$:

$\overline{y}(s) = G(s)\,K(s)\,\overline{r}(s) = F(s)\,\overline{r}(s)$

Feedback is used to combat the effects of uncertainty.

Closed-Loop Control



For a feedback control system with demanded output $\overline{r}(s)$, input disturbance $\overline{d}_i(s)$, output disturbance $\overline{d}_o(s)$ and controlled output $\overline{y}(s)$:

$$\overline{y}(s) = \frac{1}{1+G(s)K(s)}\overline{d}_o(s) + \frac{G(s)}{1+G(s)K(s)}\overline{d}_i(s) + \frac{G(s)K(s)}{1+G(s)K(s)}\overline{r}(s)$$

$$\overline{e}(s) = \overline{r}(s) - \overline{y}(s) = -\frac{1}{1+G(s)K(s)}\overline{d}_o(s) - \frac{G(s)}{1+G(s)K(s)}\overline{d}_i(s) + \frac{1}{1+G(s)K(s)}\overline{r}(s)$$

The Closed-Loop Characteristic Equation and Poles

For a feedback system, the Closed-Loop Poles are the roots of the Closed-Loop Characteristic Equation:

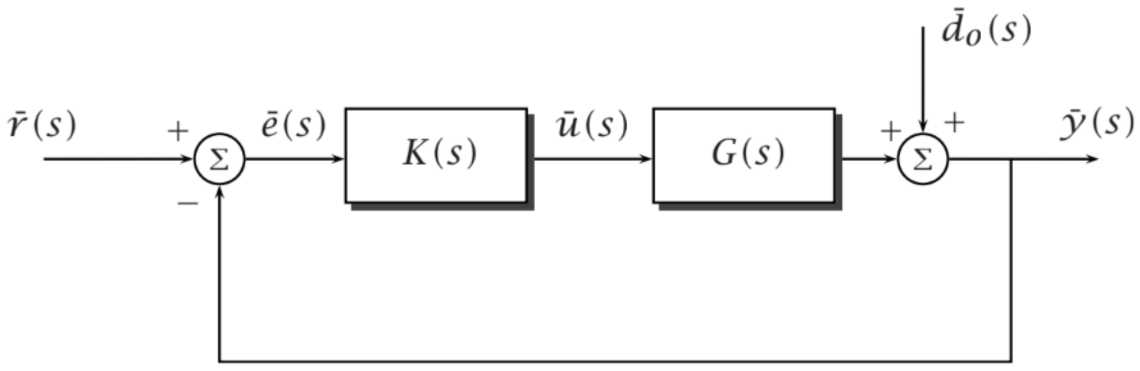
$$1 + G(s)\,K(s) = 0$$

$L(s) = G(s)\,K(s)$ is called the *Return Ratio* of the loop (and is also known as the *Loop Transfer Function*):

$$1 + L(s) = 0$$

Formally, the Return Ratio of a loop is defined as -1 times the product of all the terms around the loop.

Sensitivity and Complementary Sensitivity



For a feedback system with demanded output $\overline{r}(s)$, output disturbance $\overline{d}_o(s)$ and controlled output $\overline{y}(s)$:

$\overline{y}(s) = \frac{G(s)K(s)}{1+G(s)K(s)}\overline{r}(s) + \frac{1}{1+G(s)K(s)}\overline{d}_o(s) = \frac{L(s)}{1+L(s)}\overline{r}(s) + \frac{1}{1+L(s)}\overline{d}_o(s)$

$T(s) = \frac{L(s)}{1+L(s)}$

$S(s) = \frac{1}{1+L(s)}$

$T(s)$ is the *Complementary Sensitivity* function and $S(s)$ is the *Sensitivity* function.

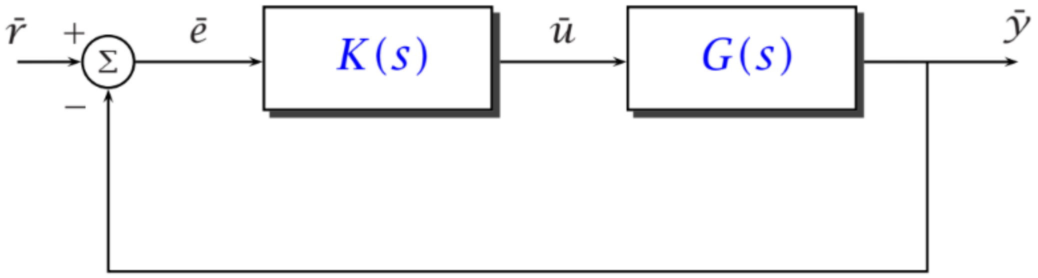
$S(s) + T(s) = 1$

The Final Value Theorem

The final value of a step response of an asymptotically stable system with transfer function $G(s)$:

$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\overline{y}(s) = \lim_{s\to 0} s\frac{G(s)}{s} = G(0)$

Steady-State Response and Error



For a feedback system with demanded output $\overline{r}(s)$ and controlled output $\overline{y}(s)$:

$\overline{y}(s) = \frac{L(s)}{1+L(s)}\overline{r}(s)$

$\overline{e}(s) = \frac{1}{1+L(s)}\overline{r}(s)$

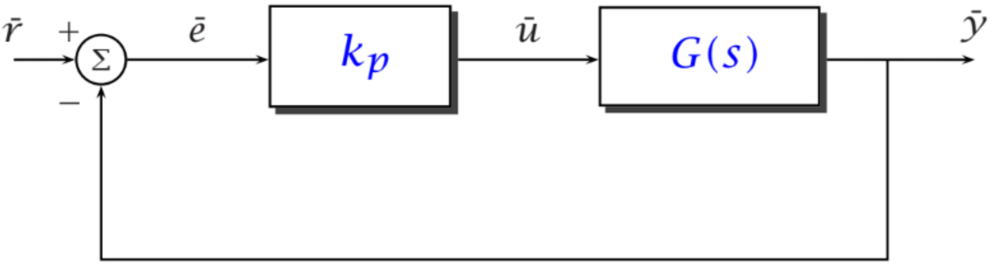
For a step demand, $r(t) = H(t)$:

$\lim_{t\to\infty} y(t) = \left. s \times \frac{L(s)}{1+L(s)} \times \frac{1}{s}\right|_{s=0} = \frac{L(0)}{1+L(0)}$

$\lim_{t\to\infty} e(t) = \left. s \times \frac{1}{1+L(s)} \times \frac{1}{s}\right|_{s=0} = \frac{1}{1+L(0)}$

Proportional Control

$K(s) = k_p$



Increasing k_p results in increased accuracy of control, but increased control action, reduced damping and possible loss of closed-loop stability for large k_p.

For control system with $G(s) = \frac{1}{(s+1)^2}$:

$\overline{y}(s) = \frac{k_p}{s^2+2s+1+k_p}\overline{r}(s)$

$\overline{e}(s) = \frac{(s+1)^2}{s^2+2s+1+k_p}\overline{r}(s)$

Closed-loop poles at $s = -1 \pm j\sqrt{k_p}$.

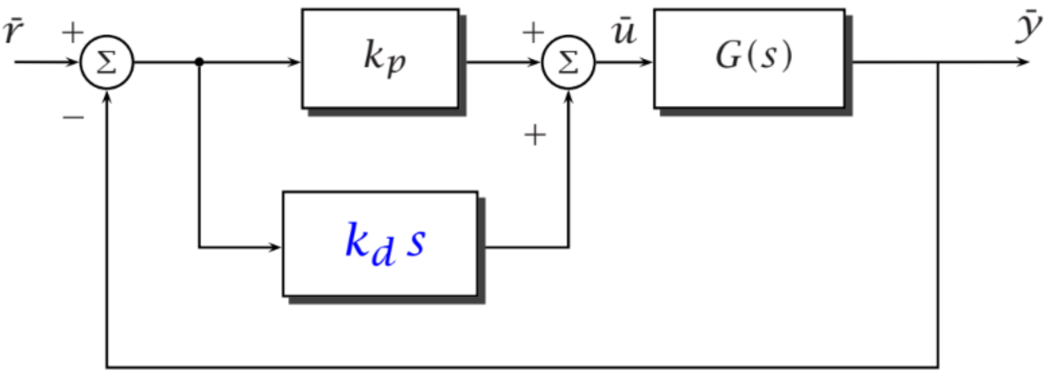
$\lim_{t\to\infty} y(t) = \frac{k_p}{1+k_p}$

$\lim_{t\to\infty} e(t) = \frac{1}{1+k_p}$

Hence, increasing k_p gives smaller steady-state errors but a more oscillatory transient response.

Proportional + Derivative (PD) Control

$K(s) = k_p + k_d s$



Increasing k_d results in increased damping and greater sensitivity to noise.

For control system with $G(s) = \frac{1}{(s+1)^2}$:

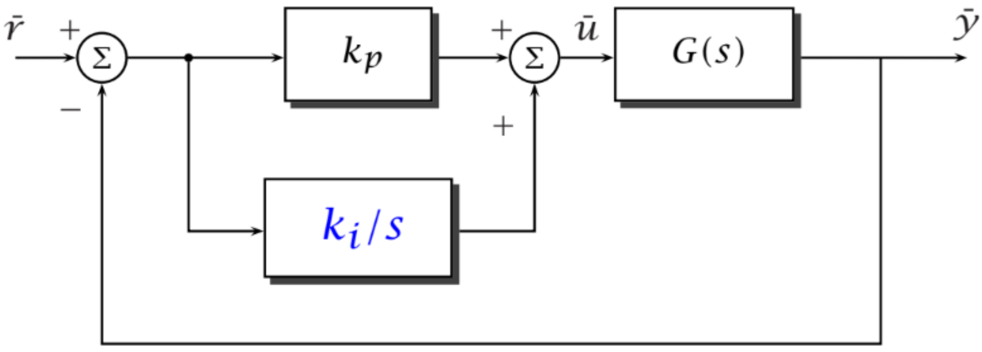
$\overline{y}(s) = \frac{k_p+k_d s}{s^2+(2+k_d)s+1+k_p}\overline{r}(s)$

$\omega_n = \sqrt{1+k_p}$

$\zeta = \frac{2+k_d}{2\sqrt{1+k_p}}$

Proportional + Integral (PI) Control

$K(s) = k_p + \frac{k_i}{s}$



For control system with $G(s) = \frac{1}{(s+1)^2}$:

$\bar{y}(s) = \frac{k_p s + k_i}{s(s+1)^2 + k_p s + k_i} \bar{r}(s)$

$\bar{e}(s) = \frac{s(s+1)^2}{s(s+1)^2 + k_p s + k_i} \bar{r}(s)$

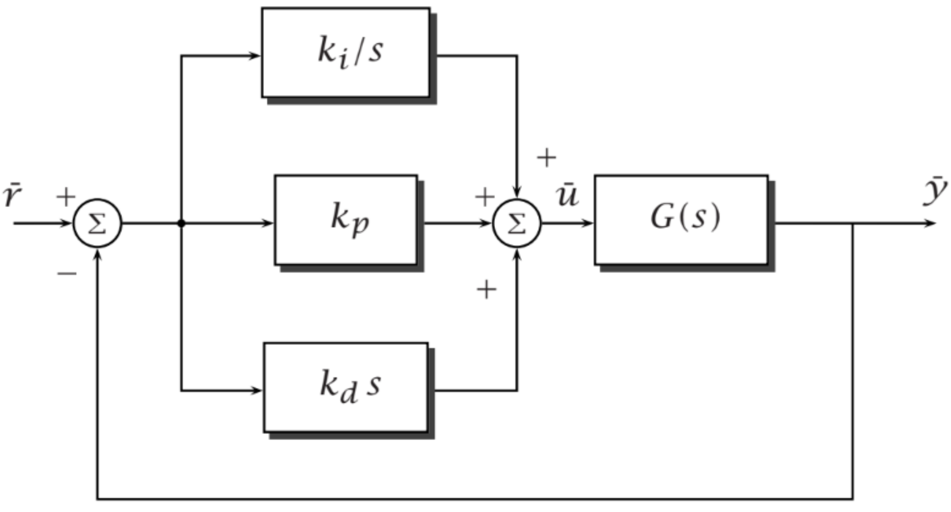
For $r(t) = H(t)$:

$\lim_{t \to \infty} y(t) = 1$

$\lim_{t \to \infty} e(t) = 0$

Proportional + Integral + Derivative (PID) Control

$K(s) = k_p + \frac{k_i}{s} + k_d s$



This can potentially combine the advantages of both derivative and integral action but can be difficult to tune.

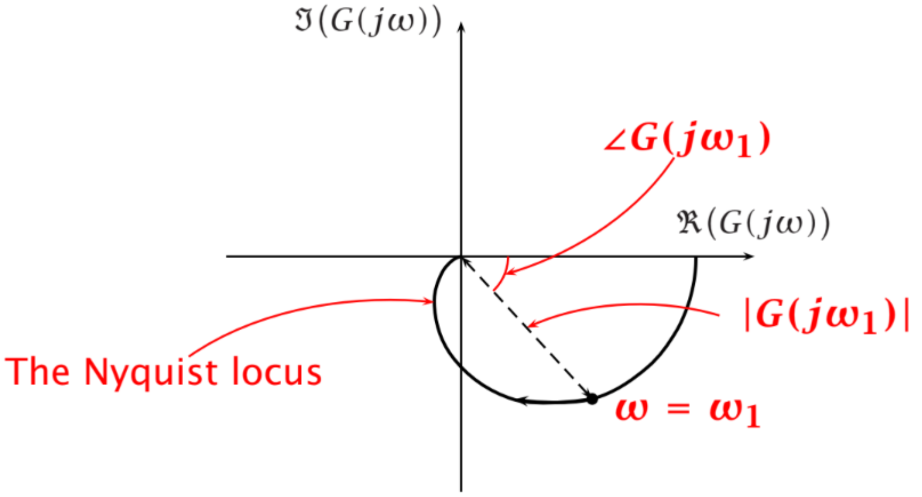
Feedback Stability and the Nyquist Diagram

Feedback Stability

Nyquist's Stability Theorem allows us to deduce closed-loop properties: the location of the poles of $\frac{L(s)}{1+L(s)}$, from open-loop properties: frequency response of the return ratio $L(j\omega)$.

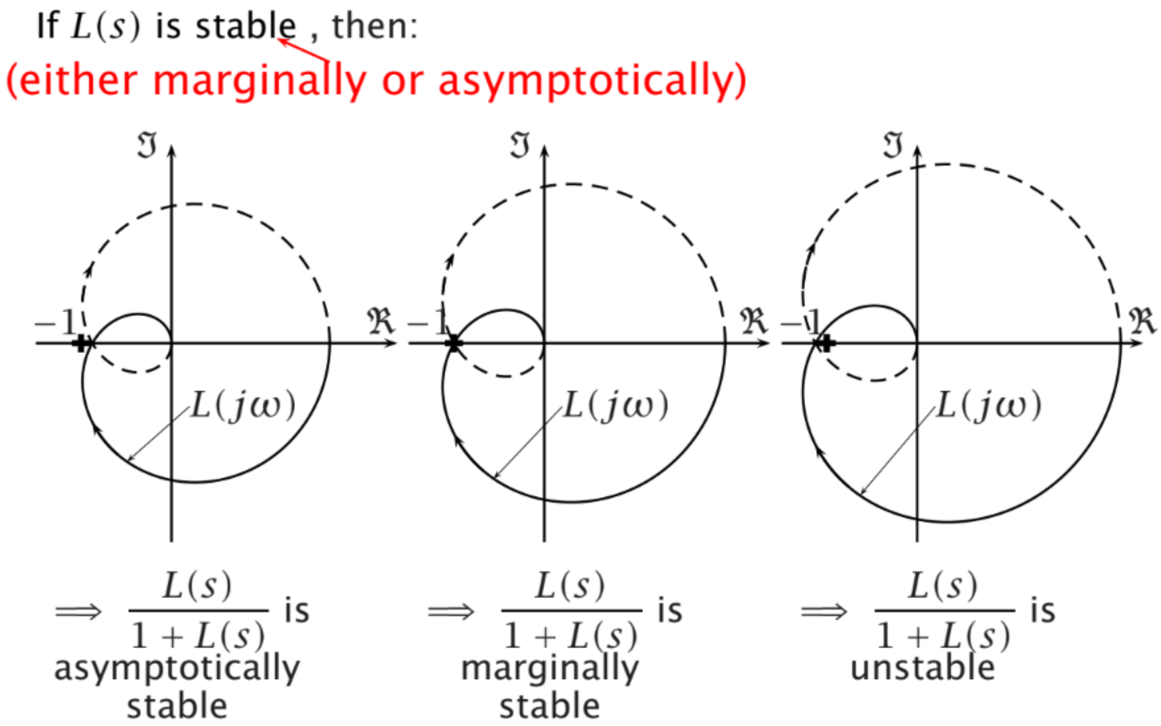
If the Nyquist locus passes through the point '-1', then the closed-loop frequency response becomes **infinite** at that frequency and there is a sustained oscillation of the feedback system even when there is no external signal.

Nyquist Stability Theorem



The Nyquist Stability Theorem:

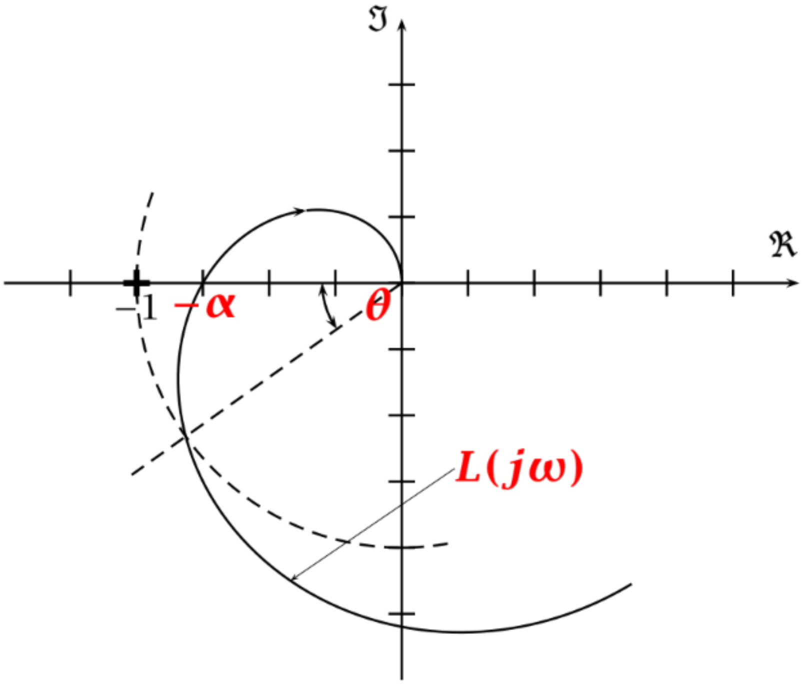
> For a feedback system with an asymptotically stable return ratio $L(s)$. In this case, the feedback system is asymptotically stable if and only if the point $-1 + j0$ is not encircled by the 'full' Nyquist diagram of $L(j\omega)$, for $-\infty < \omega < \infty$.



Gain and Phase Margins

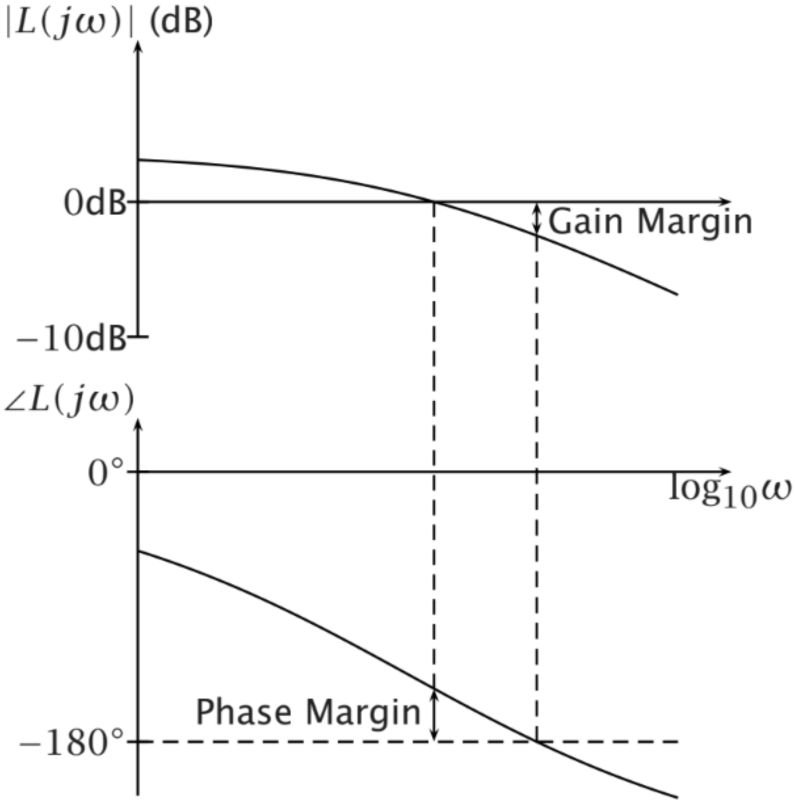
The **gain margin** measures how much the gain of the return ratio can be increased before the closed-loop system becomes unstable.

The **phase margin** measures how much phase lag can be added to the return ratio before the closed-loop system becomes unstable.



$$\text{Gain Margin} = \frac{1}{\alpha} \qquad \text{Phase Margin} = \theta$$

Gain and phase margins can also be obtained from the Bode plot:



Gain Margin = $20\log_{10}4/3 = 2.5$dB. Phase Margin = $35°$ (as before)