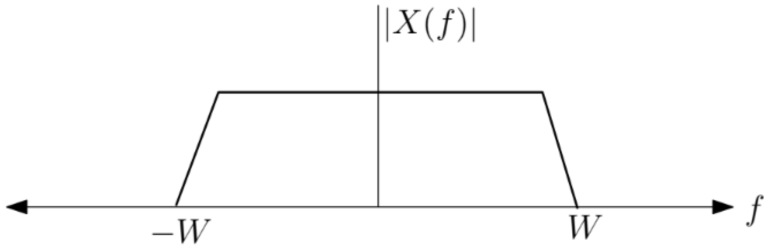
Signals and Channels

Bandwidth

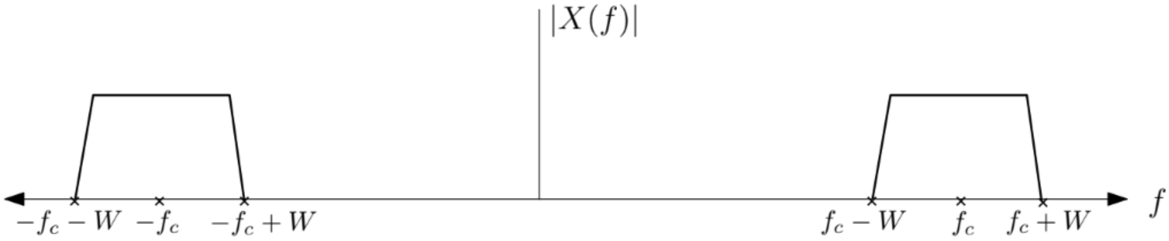
The bandwidth of a signal is roughly the range of frequencies over which its spectrum (Fourier transform) is non-zero.



Many real-world signals are time-limited and there will not be **strictly limited** in frequency.

Passband Signals

A signal is said to be **passband** if its spectral content is centred around $\pm f_c$ where $f_c \gg 0$.



Communication Channels

Channel is the medium used to transmit the signal from transmitter to receiver. It introduces **attenuation** and **noise** which can cause **errors** at the receiver.

Modelling a Channel

Channels are often modelled as **linear systems** with additive noise.

$y(t) = h(t) * x(t) + n(t)$

$Y(f) = H(f) X(f) + N(f)$

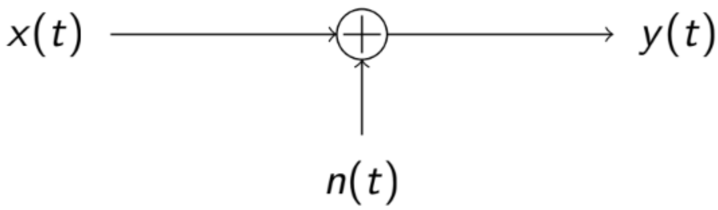
If the input is restricted to the band where the channel $H(f)$ is flat, then the channel is:

$y(t) = x(t) + n(t)$

$Y(f) = H(f) + N(f)$

Additive Gaussian Noise

Thermal noise $n(t)$ is modelled as a Gaussian random process where at each time t, $n(t)$ is a Gaussian random variable.



Analogue Modulation

Modulation

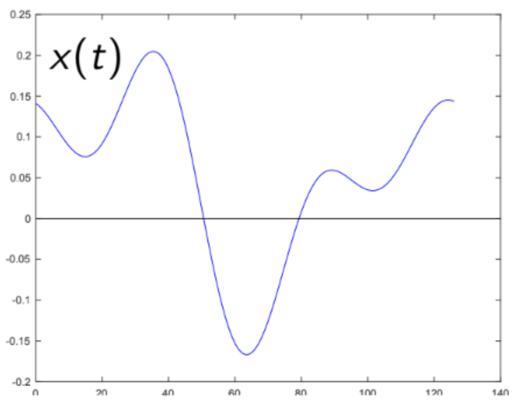
Modulation is the process by which some characteristics of a carrier wave is varied in accordance with an information bearing signal.

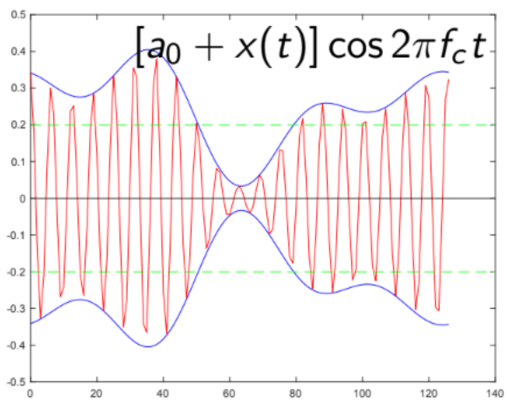
Analogue modulation is when a **continues information signal** $x(t)$ is used to directly modulate the carrier wave.

Amplitude Modulation (AM)

Transmitted AM signal is:

$s_{\text{AM}}(t) = [a_0 + x(t)]\cos(2\pi f_c t)$

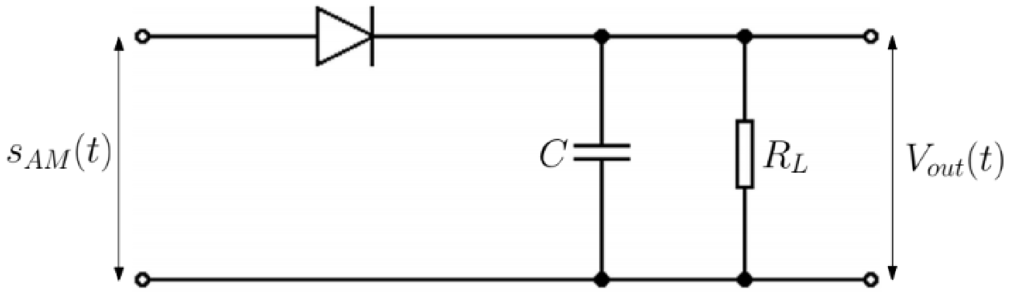




The **modulation index** of the AM signal is defined as the percentage that the carrier's amplitude varies above and below its unmodulated level:

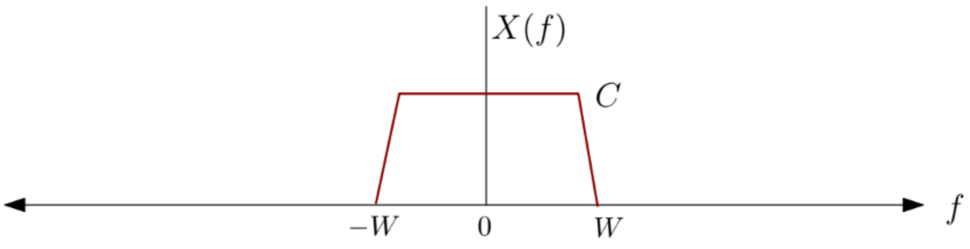
$m_A = \frac{\max_t |x(t)|}{a_0}$

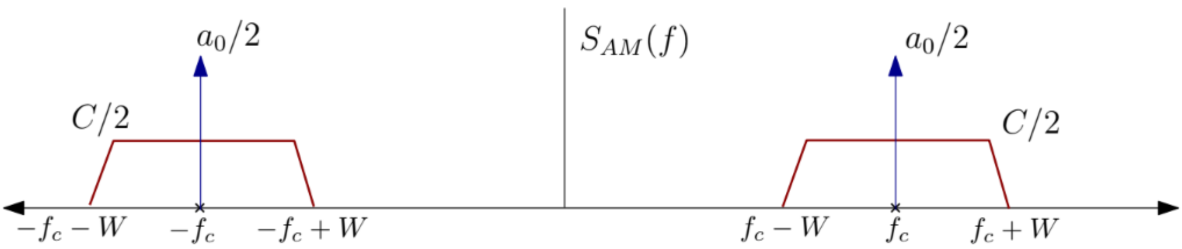
AM receiver is an envelope detector.



Spectrum of AM

$S_{\mathrm{AM}}(f) = \frac{a_0}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{1}{2}[X(f - f_c) + X(f + f_c)]$





Properties of AM

If $x(t)$ is a baseband signal with (one-sided) bandwidth W, the AM signal $s_{\mathrm{AM}}(t)$ is passband with bandwidth $2W$.

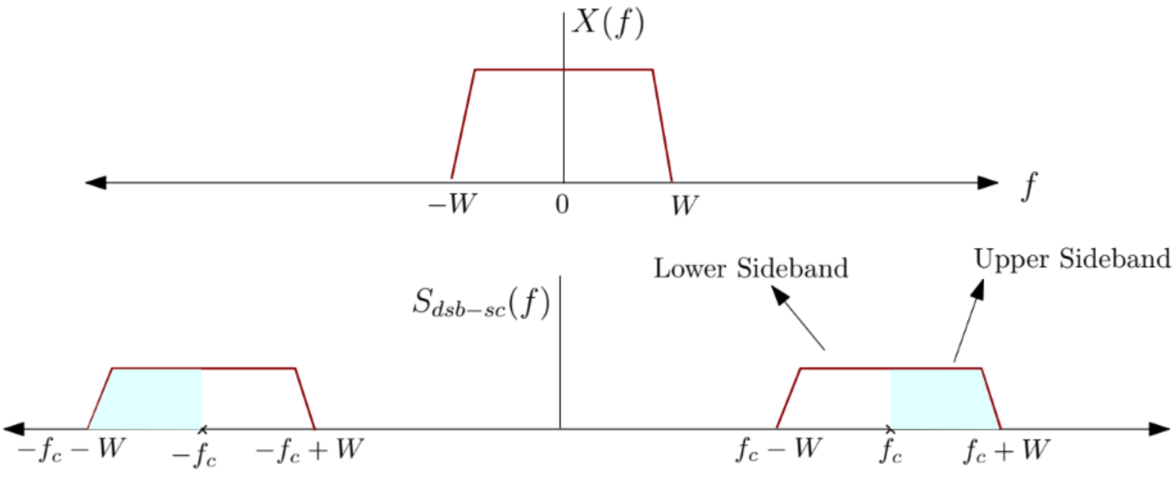
The power of the AM signal is:

$P_{\mathrm{AM}} = \frac{a_0^2}{2} + \frac{P_X}{2}$

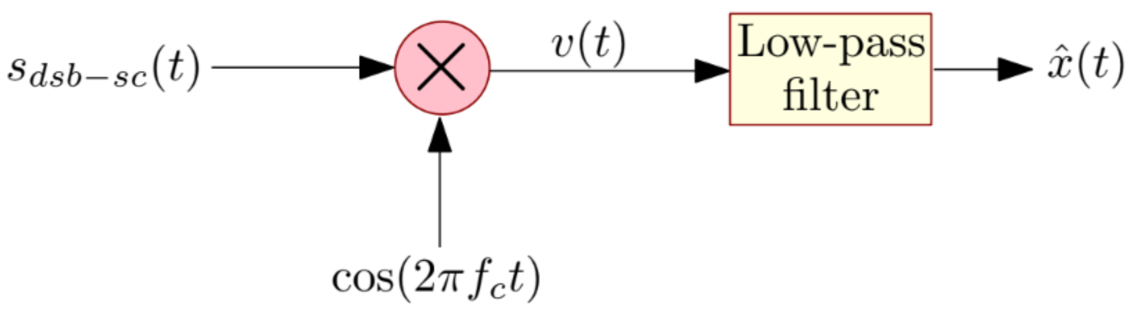
Double Sideband Suppressed Carrier (DSB-SC)

In DSB-SC, we transmit only the sidebands and suppress the carrier.

$s_{dsb-sc}(t) = x(t)\cos(2\pi f_c t)$



DSB-SC Receiver: Product modulator and low-pass filter.



Multiplying received signal by $\cos(2\pi f_c t)$:

$v\left(t\right)=x\left(t\right)\cos^2\left(2\pi f_c t\right)=\frac{x(t)}{2}+\frac{x(t)\,\cos(4\pi f_c t)}{2}$

Low-pass filter eliminates the high-frequency component.

Properties of DSB-SC

$S_{dsb-sc}\left(f\right)=\frac{1}{2}[X\left(f+f_c\right)+X\left(f-f_c\right)]$

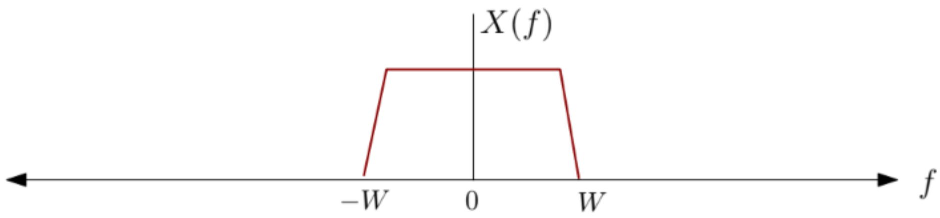
Bandwidth of DSB-SC is same as AM.

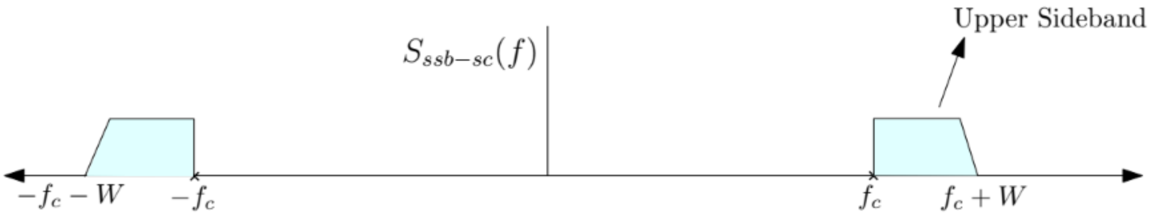
$B_{dsb-sc}=2W$

DSB-SC requires less power than AM as the carrier is not transmitted.

$P_{dsb-sc}=\frac{P_X}{2}$

Single Sideband Suppressed Carrier (SSB-SC)





For real $x(t)$,
$X(-f) = X^*(f)$.

Bandwidth of SSB-SC is half of that of AM or DBS-SC.

$B_{ssb-sc} = W$

Power is half of DSB-SC.

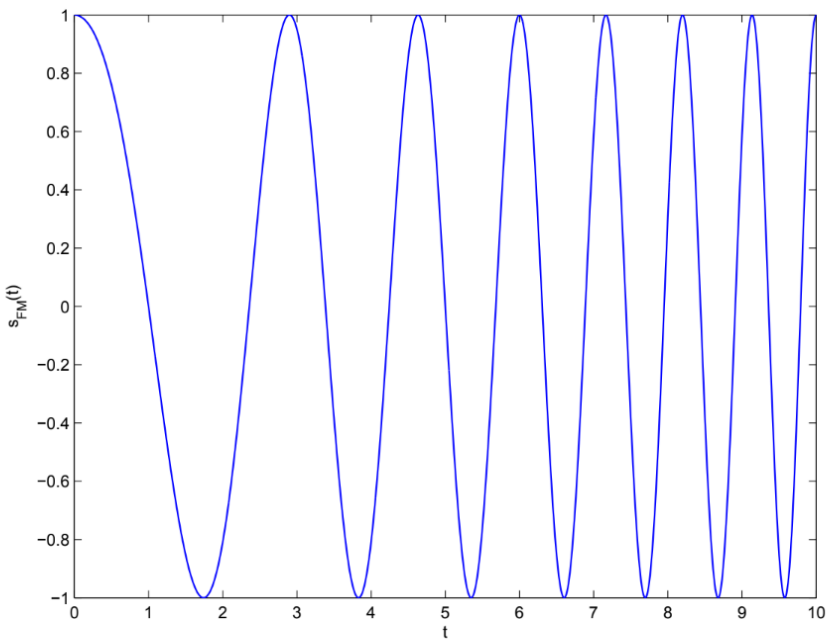
$P_{ssb-sc} = \frac{P_X}{4}$

Frequency Modulation (FM)

In FM, the information signal $x(t)$ modulates the **instantaneous frequency** of the carrier wave.

$f(t) = f_c + k_f x(t)$

$s_{\text{FM}}(t) = A_c \cos\left(2\pi \int_0^t f(t)\,\text{du}\right) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t x(u)\,\text{du}\right)$



FM demodulator is a differentiator with an envelope detector.

$\frac{ds_{\text{FM}}(t)}{\text{dt}} = -2\pi A_c \left[f_c + k_f x(t)\right] \sin\left(2\pi f_c t + 2\pi k_f \int_0^t x(u)\,\text{du}\right)$

Properties of FM

$P_{\mathrm{FM}} = \frac{A_c^2}{2}$

FM is more robust to additive noise than AM but at the cost of increased transmission bandwidth.

Spectrum of FM Signal

For a tone $x(t) = a_x \cos(2\pi f_x t)$:

$\theta(t) = 2\pi f_c t + \frac{k_f a_x}{f_x} \sin(2\pi f_x t)$

$s_{\mathrm{FM}}(t) = A_c \cos(2\pi f_c t + +\beta \sin(2\pi f_x t))$

$\Delta f = k_x a_x$ is the **frequency deviation** which is the maximum deviation of the carrier frequency $f(t)$ from f_c.

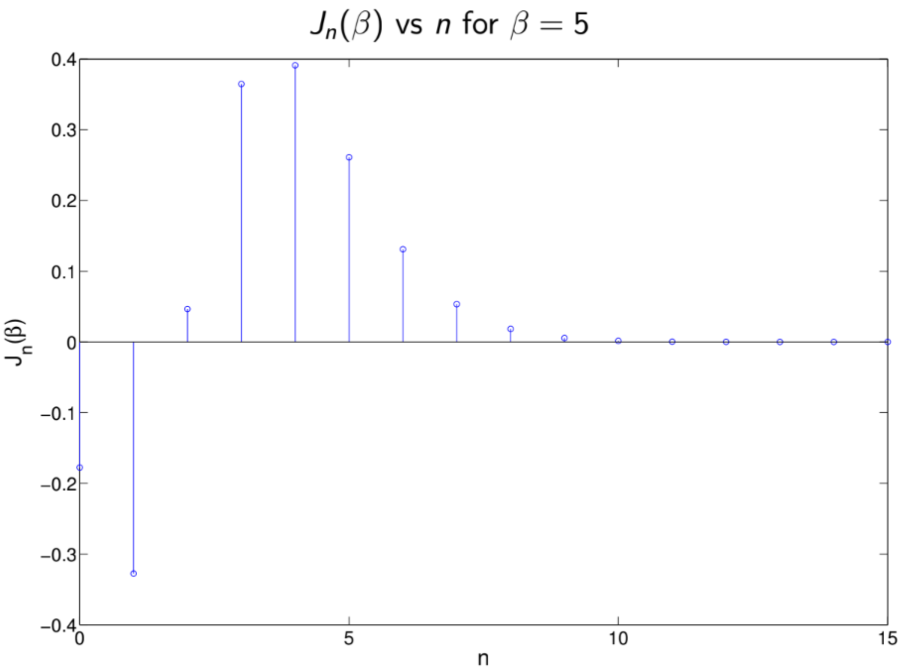
$\beta = \frac{k_f a_x}{f_x} = \frac{\Delta f}{f_x}$

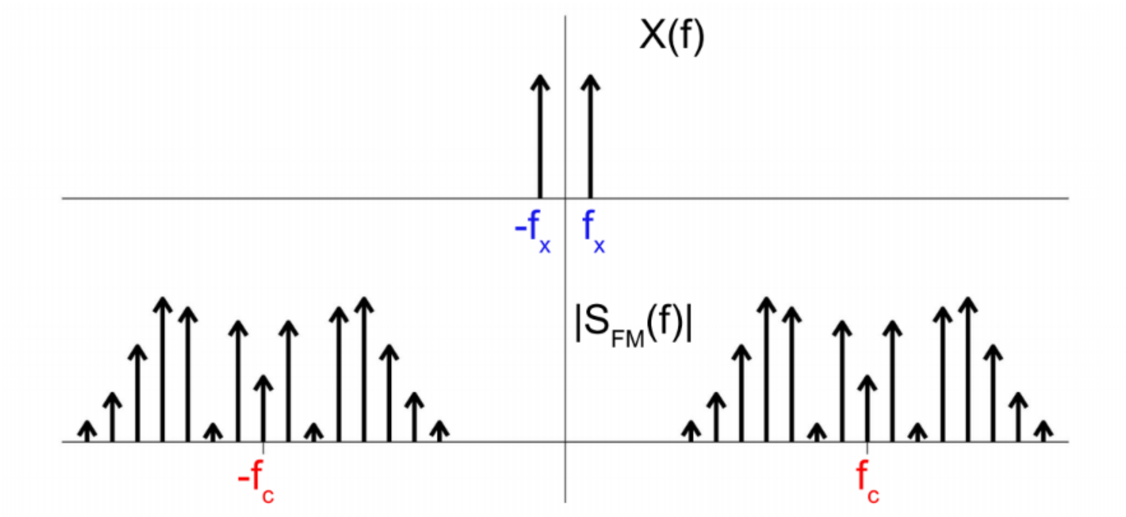
β is the **modulation index** which is the maximum deviation of the carrier phase $\theta(t)$ from $2\pi f_c t$.

$S_{\mathrm{FM}}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - n f_x) + \delta(f + f_c + n f_x)\right]$

$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin u - nu)} \mathrm{du}$

$J_n(.)$ Is called the n-th order Bessel function of the first kind.





Bandwidth of FM Signals

Carson's rule for the **effective bandwidth** of FM signals:

The bandwidth of an FM signal generated by modulating a single tone is:

$B_{\text{FM}} \approx 2\Delta f + 2f_x = 2\Delta f\left(1 + \frac{1}{\beta}\right)$

For an FM signal generated by modulating a general signal $x(t)$ with bandwidth W:

$B_{\text{FM}} \approx 2\Delta f + 2W$

Digitization of Analogue Signals

Types of Sources

Analogue source is a continuous-time, continuous-amplitude source.

Digital source is a discrete-time sequence of symbols drawn from a finite alphabet.

Digitization of Analogue Signals

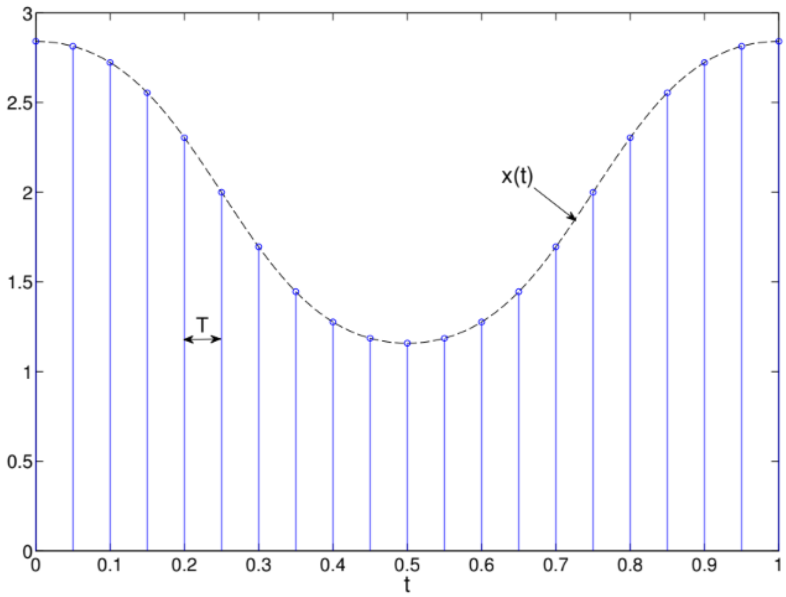
Digitization is the process by which an analogue (continuous) signal is converted into digital (discrete) format. It consists of
sampling, which discretizes the time axis, and **quantization**, which discretizes the signal amplitude axis.

Digitization is also called analogue-to-digital conversion (ADC).

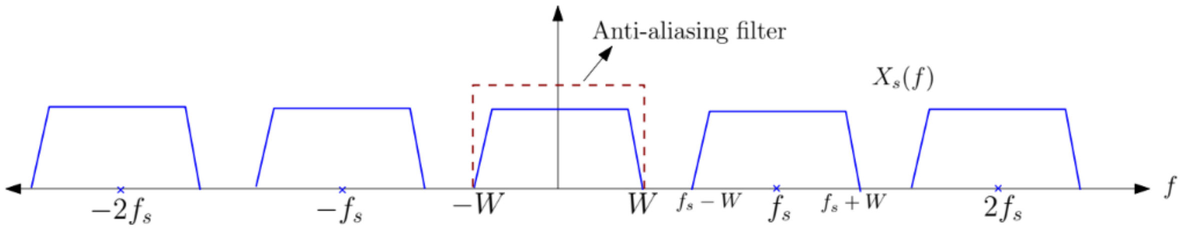
Sampling

$x_s(t) = \sum_{n=-\infty}^{\infty} x(\text{nT})\,\delta(t - nT) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)\,e^{j\frac{2\pi n}{T}t}$

$X_s(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X\left(f - \frac{n}{T}\right)$



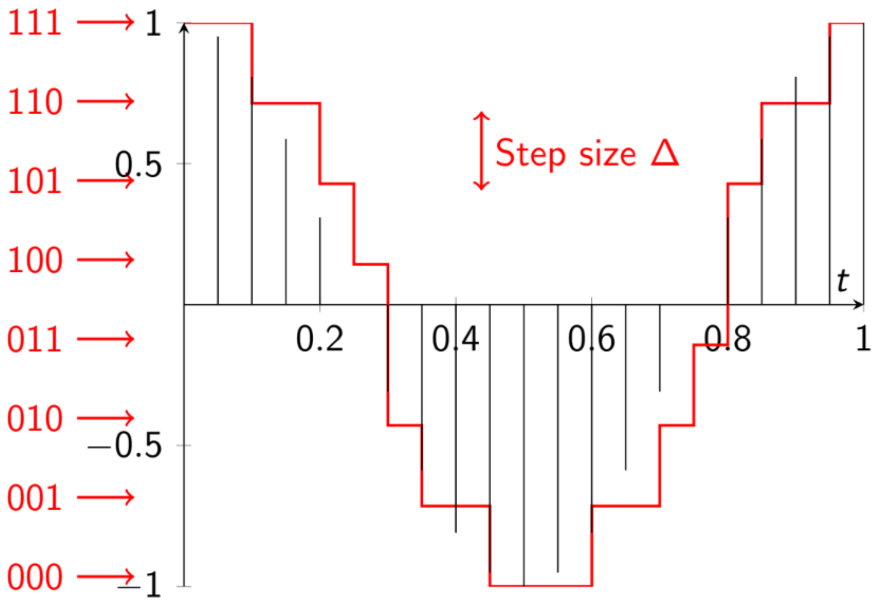
X_f can be recovered from $X_s(f)$ using an ideal reconstruction or **anti-aliasing** filter if $f_s > 2W$.



Uniform Quantization

The sampled signal can take continuous values which is converted digital by assigning a discrete amplitude from a finite set of levels with step Δ and assigning bits to those amplitudes.

Each sample $x(\mathrm{nT})$ is mapped to the nearest quantization level.



Sampling is a **lossless** procedure as long as the sampling rate is greater than the Nyquist rate while quantization is **always lossy**.

Quantization Noise as a Random Variable

$e_Q(z) = z - Q(z)$

e_Q is modelled as a random variable uniformly distributed in $\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$, and noise power is computed as:

$N_Q = E\left[e_Q^2\right] = \int_{-\Delta/2}^{\Delta/2} u^2 \frac{1}{\Delta} \mathrm{du} = \frac{\Delta^2}{12}$

Signal to Quantization Noise ratio

If the signal to be quantized is a sinusoid:

$SNR = \frac{\text{signal\ power}}{\text{noise\ power}} = \frac{(\text{RMS\ signal})^2}{(\text{RMS\ noise})^2} = \frac{V^2/2}{\Delta^2/12}$

For a n-bit uniform quantize with 2^n levels and step size $\Delta = \frac{2V}{2^n}$:

$SNR = 3 \times 2^{2n-1} = (1.76 + 6.02n)\ \mathrm{dB}$

Data Rate of the Digital Source

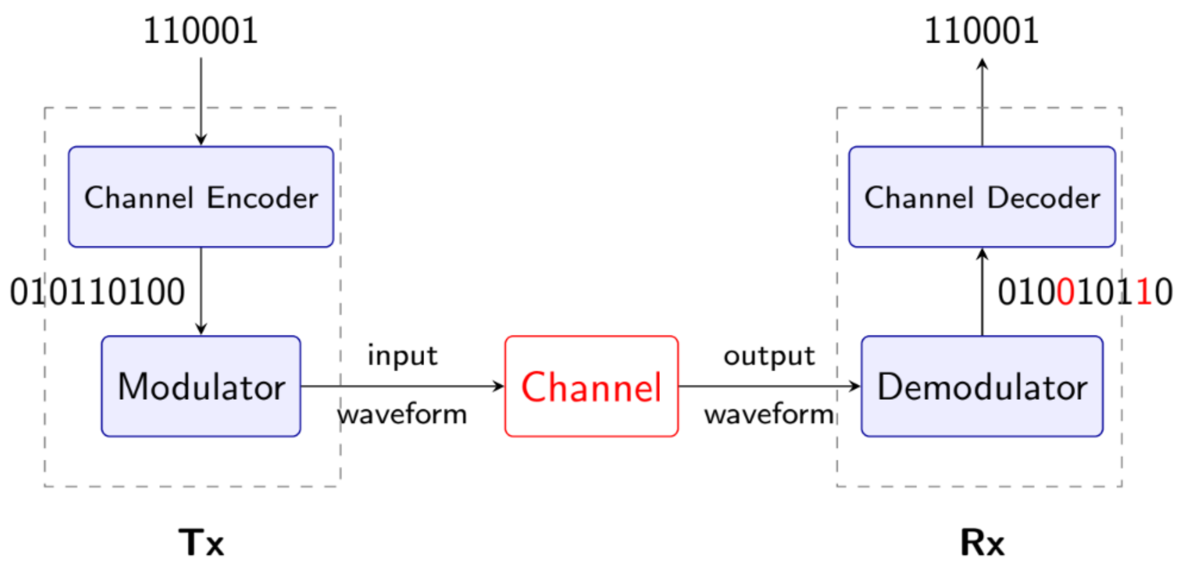
The digitized source have a rate $R = n2W$ bits/second.

Non-Uniform Quantization

Smaller step sizes in the vicinity of frequently occurring signal values, larger steps for the rarer values.

Digital Baseband Modulation

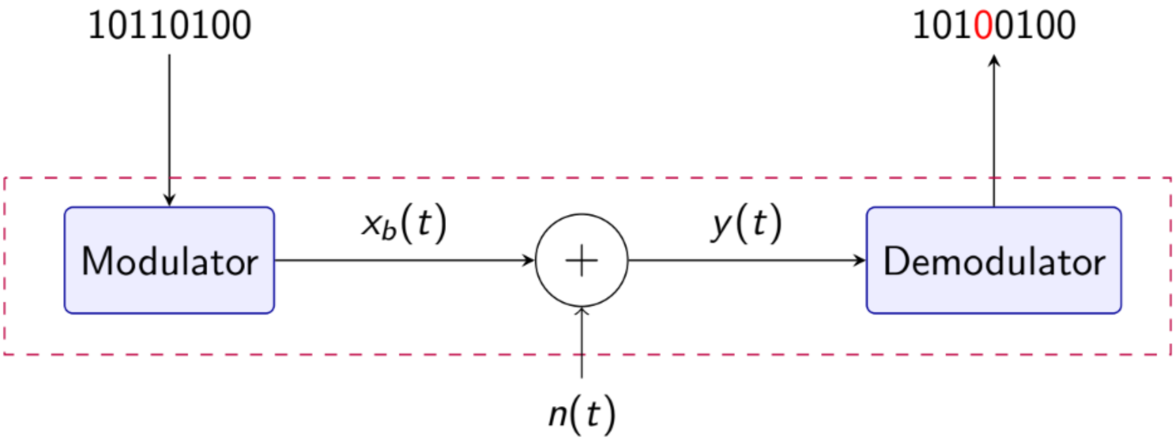
Data Transmission



The transmitter (Tx) does **encoding** to add redundancy to the sources bits to protect against noise, and **modulation** to transform the coded bits into waveforms.

The receiver (Rx) doe **demodulation** to transform noisy output waveform into output bits and **decoding** to correct errors in the
output bits and recover the source bits.

Pulse Amplitude Modulation (PAM)



The first component of the digital modulation scheme is a mapping from bits to real or complex numbers. The set of values the bits are mapped to is called the **constellation**.

In a constellation with M symbols, each symbol represents $\log_2 M$ bits.

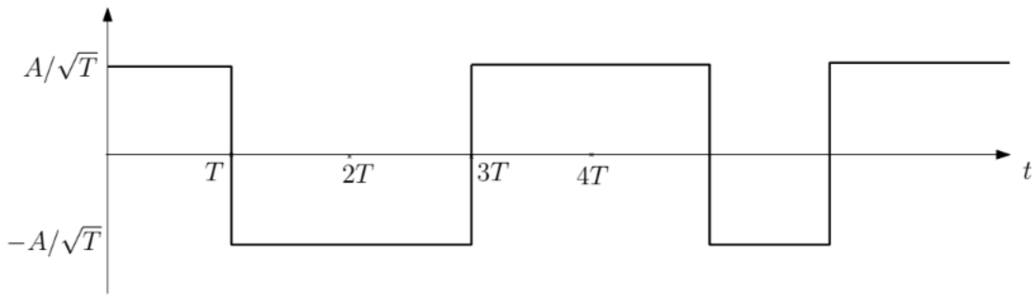
The second component is a unit-energy **baseband** waveform denoted $p(t)$ called the **pulse shape**.

$$p(t) = \frac{1}{\sqrt{T}} \text{sinc}\left(\frac{\pi t}{T}\right) \quad or \quad p(t) = \begin{cases} \frac{1}{\sqrt{T}} & for\ t \in (0, T] \\ 0 & otherwise \end{cases}$$

T is called the **symbol time** of the pulse.

A sequence of constellation symbols is used to generate a baseband signal:

$$x_b(t) = \sum_k X_k p(t - kT)$$



Rate of Transmission

The **transmission rate** is $\frac{1}{T}$ symbols/sec or $\frac{\log_2 M}{T}$ bits/second.

Desirable Properties of the Pulse Shape

$p(t)$ is chosen to decay quickly in time and be approximately band limited.

$$X_b(f) = P(f) \sum_k X_k e^{-j2\pi f k T}$$

Hence the bandwidth of $x_b(t)$ is the same as that of the pulse $p(t)$.

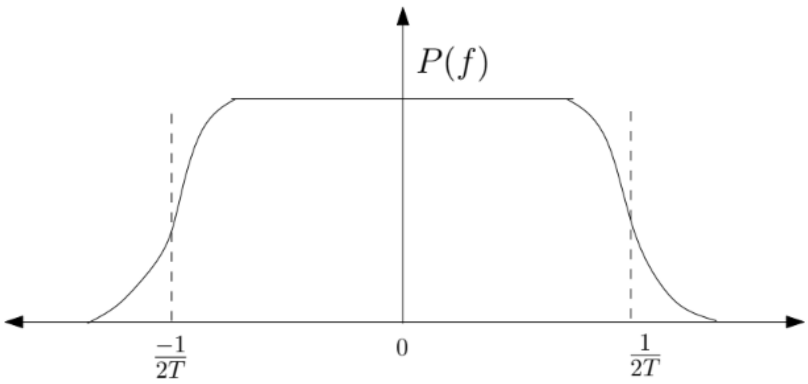
The retrieval of the information sequence from the noisy received waveform $x_b(t) + n(t)$ should be simple and relatively reliable.

Orthonormality of Pulse Shifts

$\int_{-\infty}^{\infty} p\left(t-kT\right)p\left(t-mT\right)\mathrm{dt} = \begin{cases} 1\; if\; k=m \\ 0\; if\; k\neq m \end{cases}$

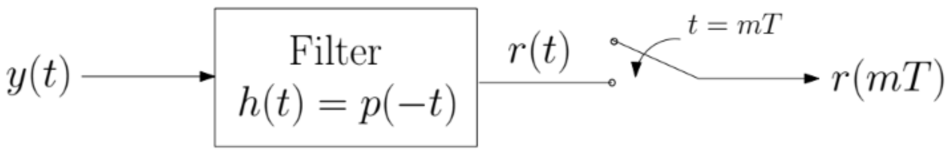
Time Decay vs. Bandwidth Trade-off

In practice, the pulse shape is often chosen to have a **root raised cosine** spectrum.



Bandwidth slightly larger than $\frac{1}{2T}$ and decay in time $|p\left(t\right)| \sim \frac{1}{|t|^2}$.

Matched Filter Demodulator



Assuming no noise:

$y\left(t\right) = x_b\left(t\right) = \sum_k X_k p\left(t-kT\right)$

$y\left(t\right)$ is passed through a filter with impulse response $h\left(t\right) = p\left(-t\right)$ called a **matched filter**.

$r\left(t\right) = x_b\left(t\right) * h\left(t\right) = \int_{-\infty}^{\infty} x_b\left(\tau\right)h\left(t-\tau\right)\mathrm{d}\tau = \int_{-\infty}^{\infty} X_k p\left(\tau-kT\right)p\left(\tau-t\right)\mathrm{d}\tau$

By sampling the filter output at time $t = mT$:

$r\left(\mathrm{mT}\right) = \int_{-\infty}^{\infty} X_k p\left(\tau-kT\right)p\left(\tau-mT\right)\mathrm{d}\tau = X_m$

Demodulation with Noise

$r\left(\mathrm{mT}\right) = X_m + \int_{-\infty}^{\infty} n\left(\tau\right)p\left(\tau-mT\right)\mathrm{d}\tau = X_m + N_m = Y_m$

Properties of the Noise

N_m is a random variable whose distribution depends on the statistics of the Gaussian random process $n\left(t\right)$.

The sequence of random variables $\left\{N_m\right\}, m = 0, 1, \ldots$ are **independent** and **identically distributed** as $\mathcal{N}\left(0, \sigma^2\right)$.

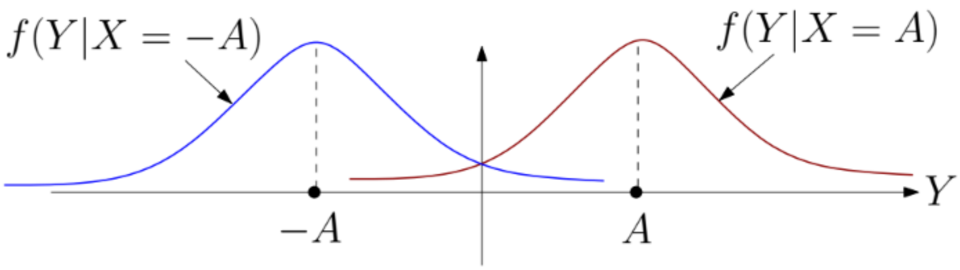
Detection for Binary PAM

For a binary PAM or BPSK (Binary Phase Shifting Key), $X_m \in \{-A, A\}$.

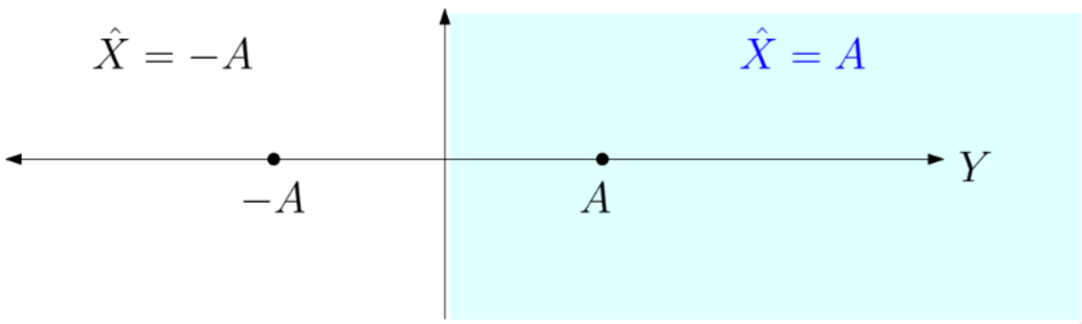
$Y = X + N$

The maximum-likelihood decoder chooses the symbol from which Y is most likely to have occurred.

$$\widehat{X} = \underset{X \epsilon A, -A}{\text{argmax}} f\left(Y|X = x\right) = \underset{X \epsilon A, -A}{\text{argmin}} (Y - X)^2$$



The detection rule partitions the space of Y (the real line) into **decision regions**.



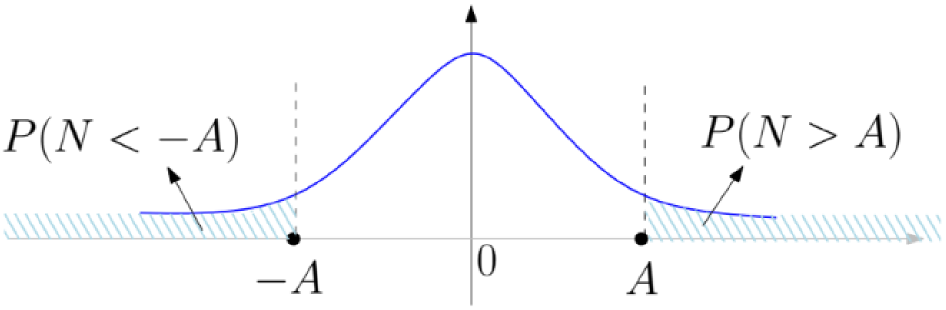
Detection for General PAM Constellations

$$\widehat{X} = \underset{X \epsilon \mathcal{C}}{\text{argmin}} (Y - X)^2$$

Choose the constellation symbol closest to the output Y.

Probability of Detection Error

$$P_e = P\left(\widehat{X} \neq X\right) = \tfrac{1}{2} P\left(\widehat{X} = A \middle| X = -A\right) + \tfrac{1}{2} P\left(\widehat{X} = -A \middle| X = A\right) = P\left(N > A\right) = P\left(\tfrac{N}{\sigma} > \tfrac{A}{\sigma}\right)$$



The error probability is usually expressed in terms of the $\mathcal{Q}$-function which is the probability that a standard Gaussian random variable takes value greater than x:

$\mathcal{Q}(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \mathrm{du} = 1 - \Phi(x)$

$P_e = \mathcal{Q}\left(\frac{A}{\sigma}\right) = \mathcal{Q}\left(\sqrt{\frac{E_s}{\sigma^2}}\right)$

E_s is the **average energy per symbol** of the constellation.

$E_s = E_b \log_2 M = \frac{1}{2}\left(A^2 + (-A)^2\right) = A^2$

E_b is the **average energy per bit** and $\frac{E_b}{\sigma^2}$ is the signal-to-noise ratio (snr) of the transmission scheme.

$P_e = \mathcal{Q}\left(\sqrt{\mathrm{snr}}\right) \approx e^{-snr/2}$

The power of the baseband PAM signal $x_b(t)$ is:

$\frac{E_s}{T} = \frac{E_b \log_2 M}{T}$

Digital Passband Modulation

Bassband to Passband

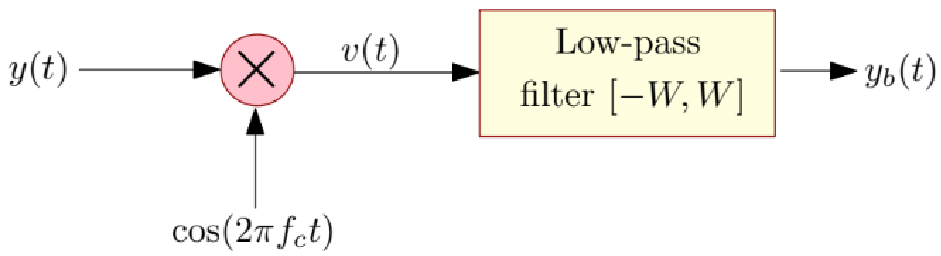
Modulate the amplitude of a high-frequency carrier with $x_b(t)$:

$x(t) = x_b(t)\cos(2\pi f_c t) = \left[\sum_k X_k p(t - kT)\right]\cos(2\pi f_c t)$

This passband modulation scheme is called **up-converted PAM**. Up-converted PAM with a rectangular pulse $p(t)$ is also called Amplitude Shift Keying (ASK).

Demodulation of Up-Converted PAM

Down-convert via product modulator and low-pass filter.



$y_b(t) = x_b(t) + n_b(t) = \sum_k X_k p(t - kT) + n_b(t)$

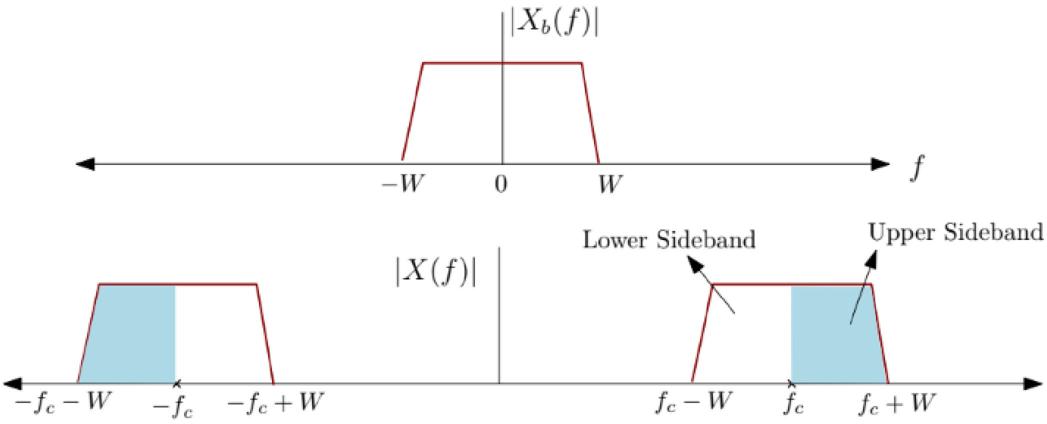
$n_b(t)$ is the baseband noise.

Demodulate **baseband** waveform by passing $y_b(t)$ through matched filter and then sample at times $\{\mathrm{mT}\}, m \in \mathbb{Z}$.

Spectrum of Up-Converted PAM

$X(f) = \frac{1}{2}\left[X_b(f - f_c) + X_b(f + f_c)\right]$

For real signal $x_b(t)$, $X_b(-f) = X_b^*(f)$. Sending both sidebands is redundant since all the information is contained in one.



Quadrature Amplitude Modulation (QAM)

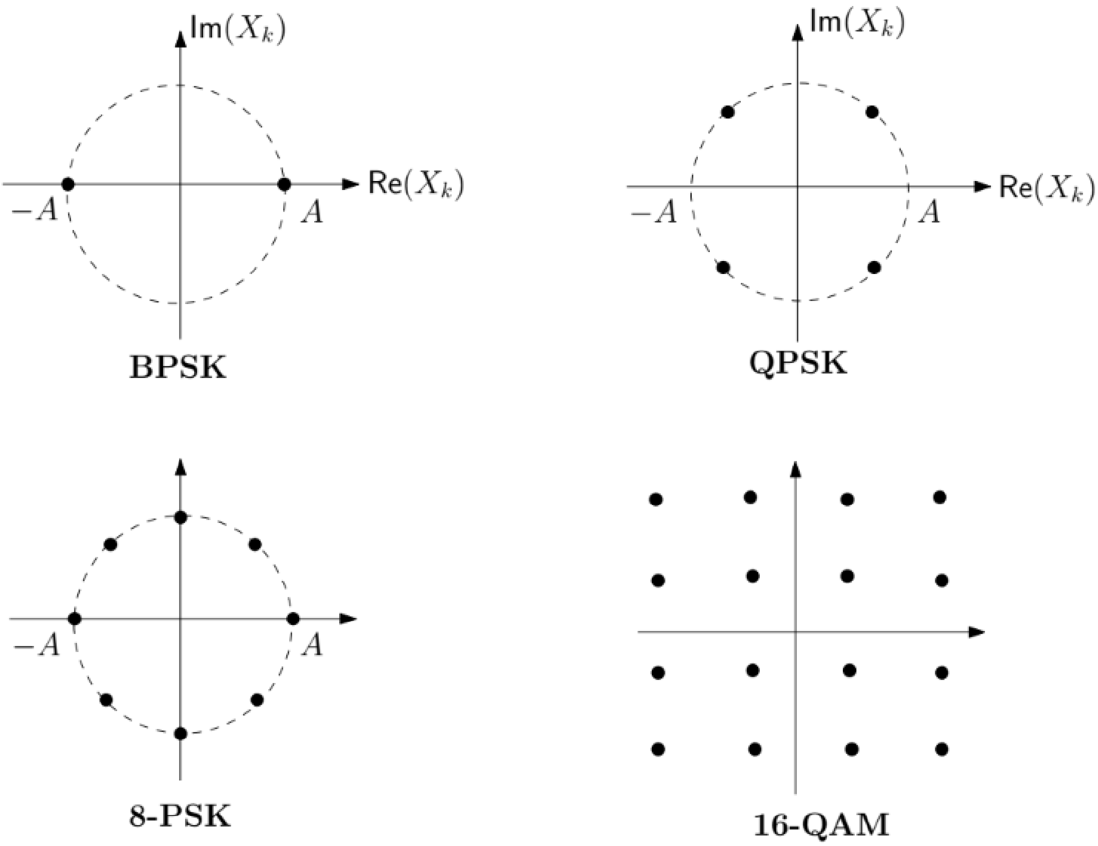
The constellation is complexed-valued, but the passband signal has to be **real**.

$x\left(t\right) = \mathrm{Re}\left[x_b\left(t\right)e^{j2\pi f_c t}\right] = \mathrm{Re}\left(x_b\left(t\right)\right)\cos(2\pi f_c t) + \mathrm{Im}\left(x_b\left(t\right)\right)\sin(2\pi f_c t)$

$x\left(t\right) = \sum_k p\left(t-kT\right)\left[\mathrm{Re}\left(X_k\right)\cos(2\pi f_c t) + \mathrm{Im}\left(X_k\right)\sin(2\pi f_c t)\right] = \sum_k p\left(t-kT\right)\left|X_k\right|\cos(2\pi f_c t + \phi_k)$

$|X_k|$ and ϕ_k are the magnitude and phase of the complex symbol X_k.

QAM Constellations



In **Phase Shifting Key** (PSK), the magnitude of X_k is constant and the information is in the phase of the symbol.

In a constellation with M symbols, each symbol corresponds to $\log_2 M$ bits.

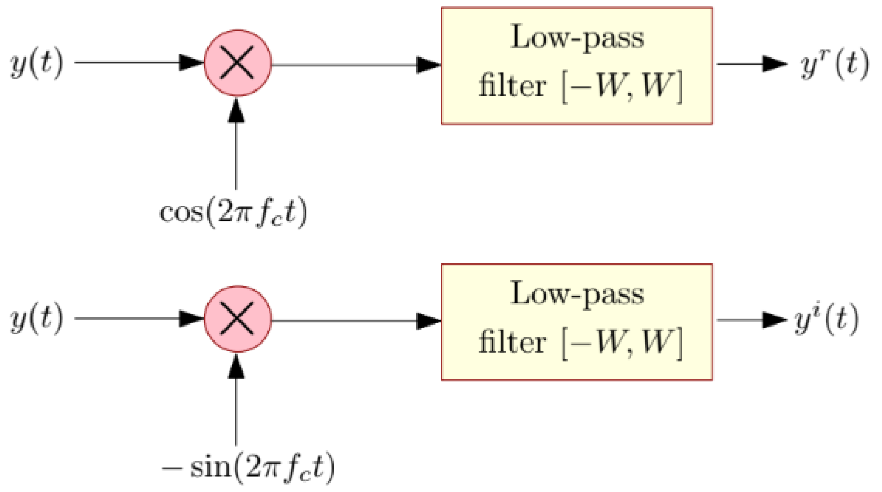
Average Energy Per Symbol

For the PSK constellations, average symbol energy $E_s = A^2$.

For 16-QAM, average energy per symbol $E_s = \frac{40d^2}{16} = 2.5d^2$. Average energy per bit is:

$E_b = \frac{E_s}{\log_2 M}$

For QAM, we need **two** product modulators, one for the cosine and the other for sine.



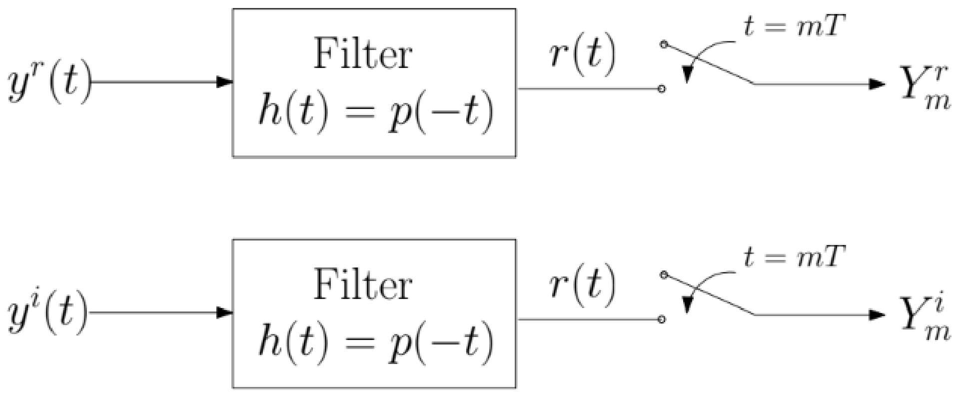
$y\,(t) = \sum_k p\,(t - kT)\left[X_k^r \cos(2\pi f_c t) - X_k^i \sin(2\pi f_c t)\right] + n\,(t)$

After down-converting,

$y^r\,(t) = \sum_k X_k^r p\,(t - kT) + n^r\,(t)$

$y^i\,(t) = \sum_k X_k^i p\,(t - kT) + n^i\,(t)$

Demodulation



The sampled outputs of the matched filters are:

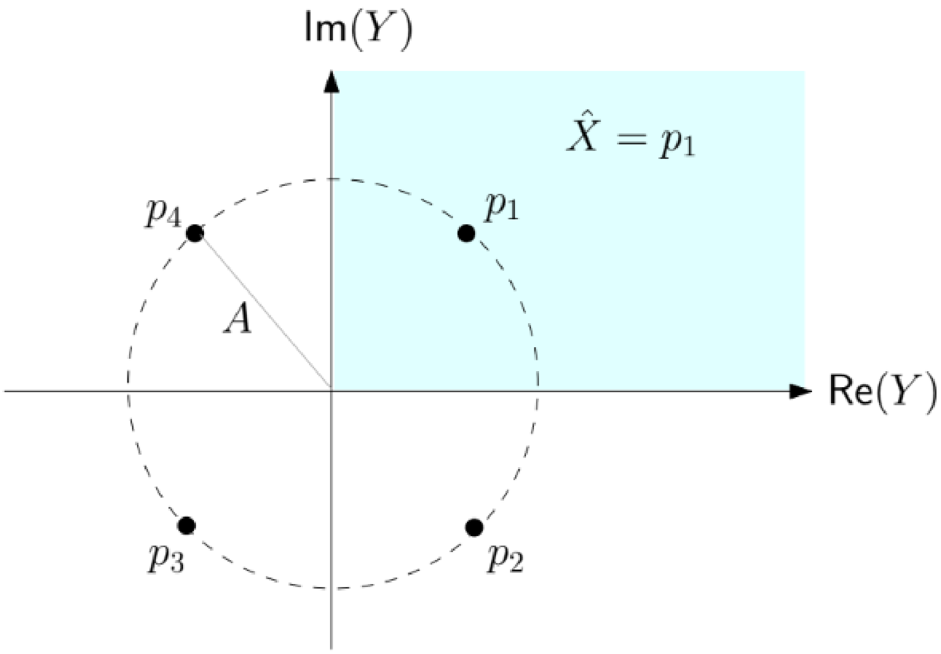
$Y_m^r = X_m^r + N_m^r$

$Y_m^i = X_m^i + N_m^i$

N_m^r and N_m^i are each independent Gaussians distributed as $\mathcal{N}\left(0, \sigma^2\right)$ for each m.

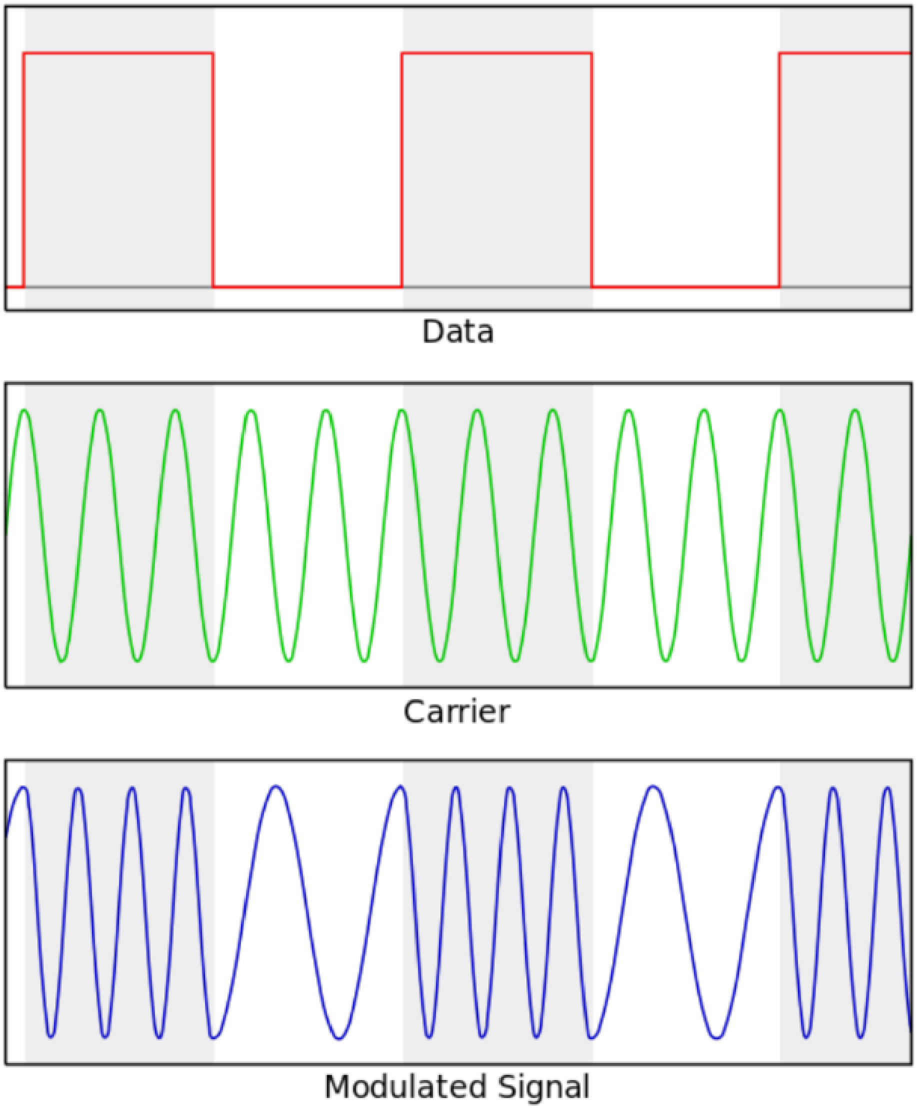
Detection

Choose the constellation symbol X closest to observed complex output Y.

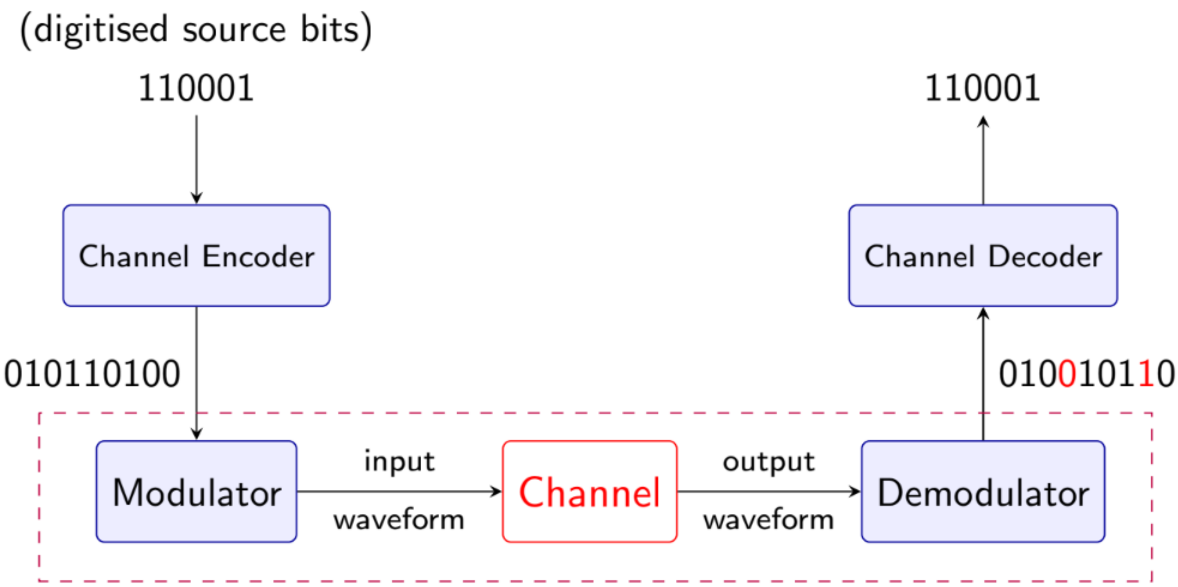


Frequency Shifting Key (FSK) modulates the frequency of a carrier to transmit digital information.

$$x(t) = \begin{cases} \cos(2\pi (f_c - \Delta_f) t) & if\ X_k = 0, \\ \cos(2\pi (f_c + \Delta_f) t) & if\ X_k = 1. \end{cases}$$

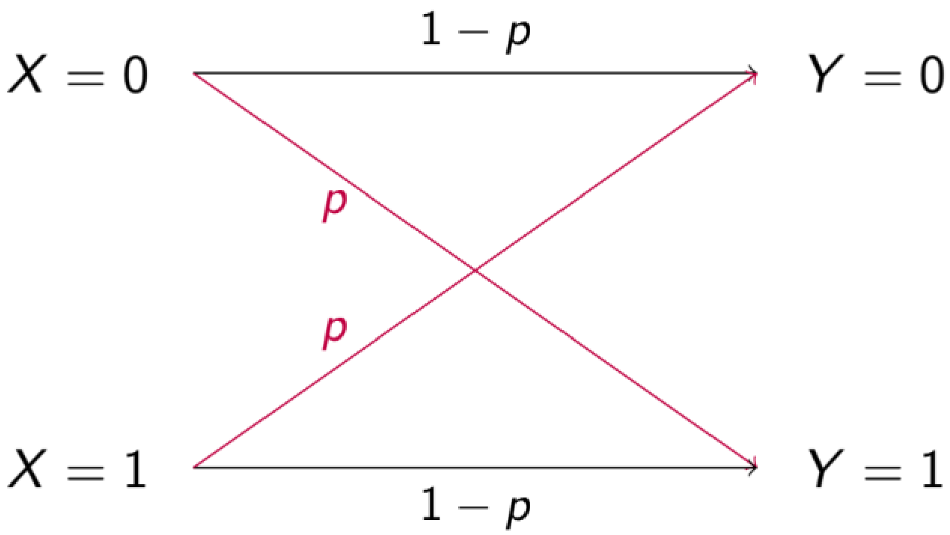


Channel Coding



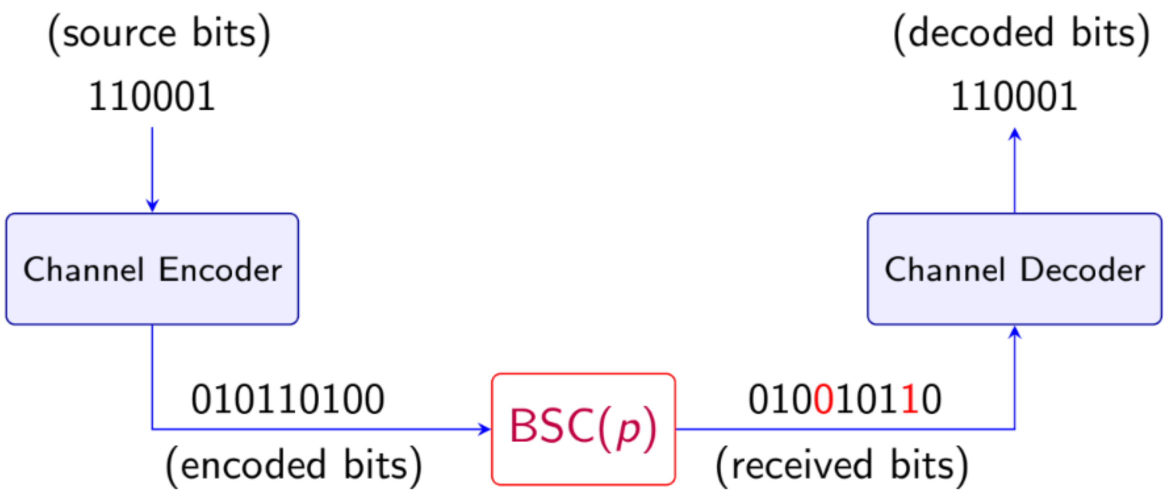
Binary Symmetric Channel (BSC)

If the modulation scheme has a **bit error probability** p, the binary channel flips each bit (0 or 1) with equal probability p.



It is called a Binary Symmetric Channel $\mathrm{BSC}\,(p)$ where p is the **crossover probability**.

Channel Coding



Channel coding consists of adding redundancy to the source bits at the transmitter to recover from errors at the receiver.

Repetition Code

$(n, 1)$ repetition code encodes by repeating each source bit n (odd) times and decodes by majority vote.

Decoding errors are data rate:

$P_e = \sum_{k=(n+1)/2}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

$R = \frac{1}{n}$

Block Code

In a block codes, every block of K source bits is represented by a sequence of N code bits called the codeword. To add redundancy,
$N > K$.

In a linear block code, the extra $N - K$ code bits are **linear functions** of the K source bits.

The rate of any (K, N) block is $\frac{K}{N}$.

The (7,4) Hamming Code

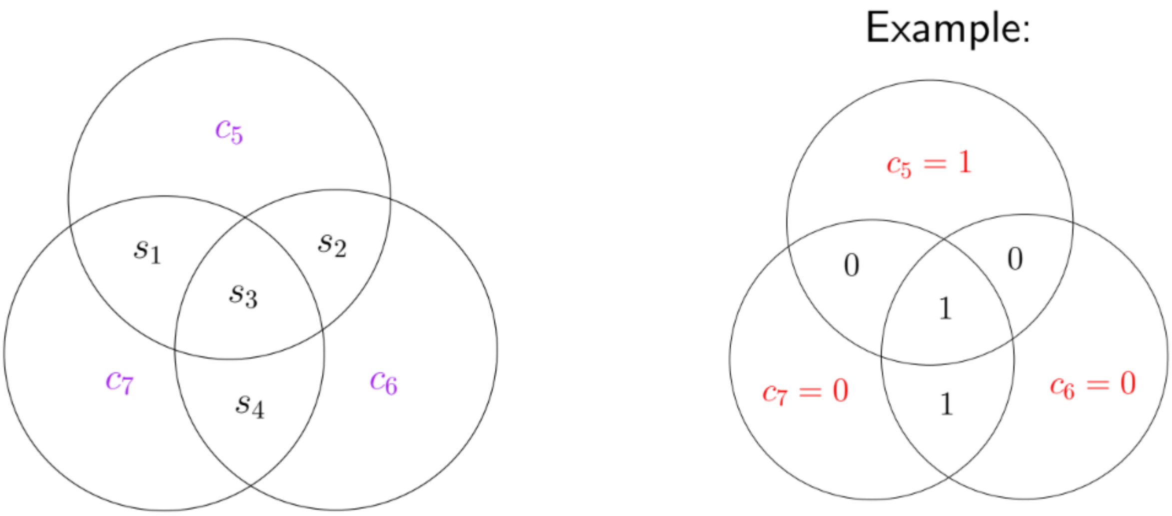
Each 4-bit source block s is encoded into 7-bit codeword c:

$c_i = s_i \ for \ i = 1,\ 2,\ 3,\ 4$

$c_5 = s_1 \oplus s_2 \oplus s_3$

$c_6 = s_2 \oplus s_3 \oplus s_4$

$c_7 = s_1 \oplus s_3 \oplus s_4$



For any Hamming codeword, the **parity** of each circle is even. If any circles have odd parity, flip exactly one bit to make all of them have even parity.

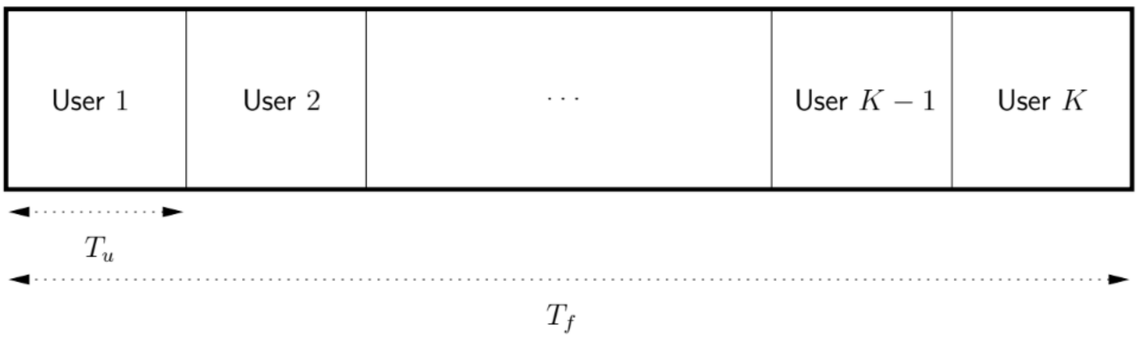
Hamming codes have a good rate but rather high probability of decoding error.

Multiple Access

Time Division Multiple Access

In **time-division** multiple access, multiple users are multiplexed in time, so that they transmit one after another using the whole bandwidth B.

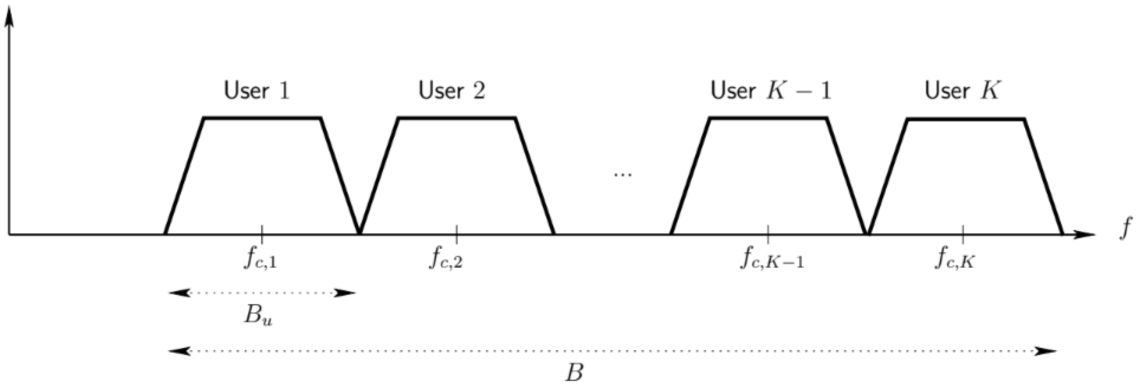
$T_u = \frac{T_f}{K}$



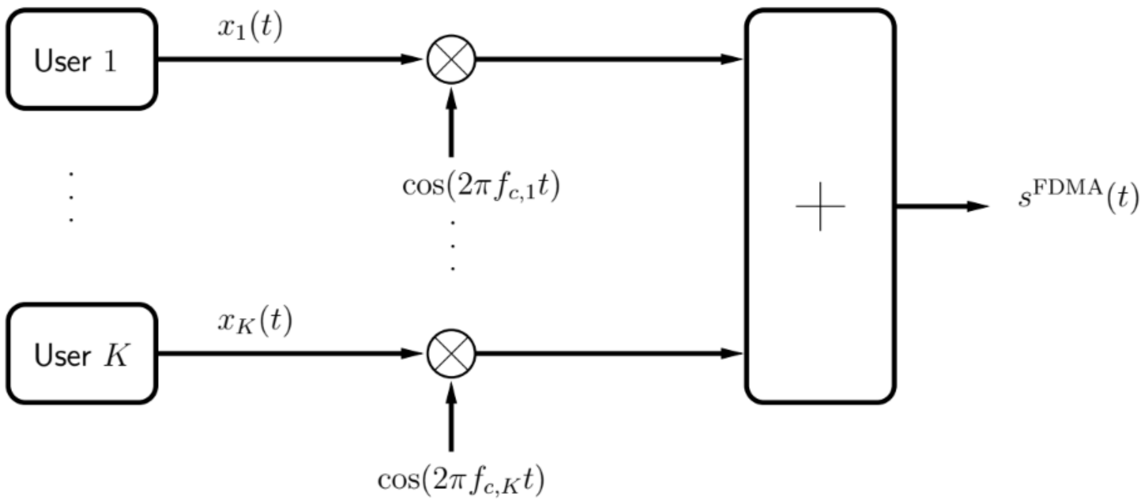
Frequency Division Multiple Access

In **frequency-division** multiple access, K users are multiplexed in the frequency domain by allocating a fraction of the total bandwidth to each one.

$B_u < \frac{B}{K}$



$s^{\mathrm{FDMA}}\left(t\right) = \sum_{i=1}^{K} x_i\left(t\right)\cos(2\pi f_{c,i}t)$

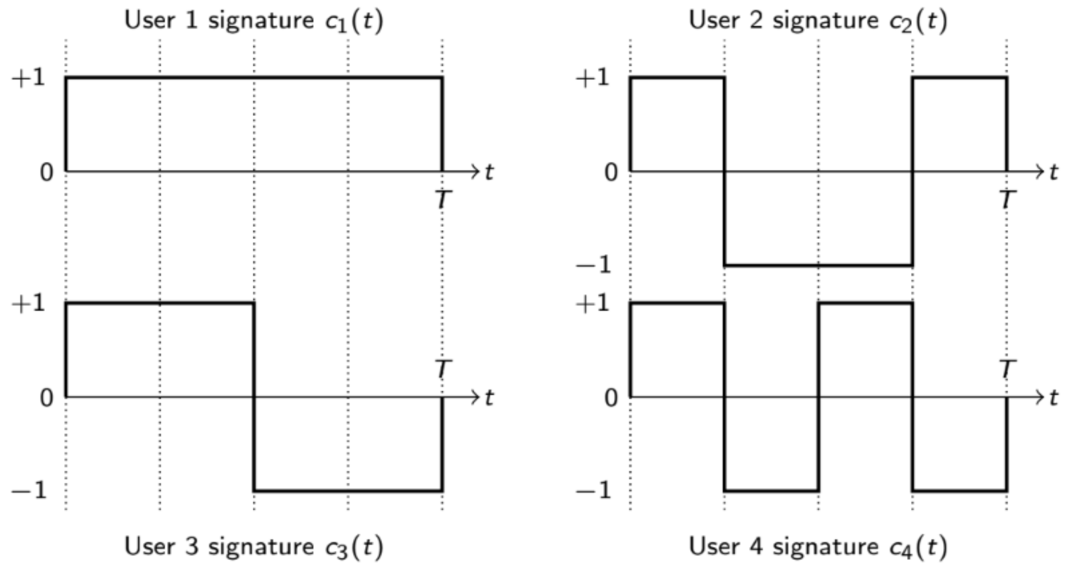


Code Division Multiple Access

In code-division multiple access, each user is given a unique signature function $c_i\left(t\right)$.

The signatures are chosen to be orthogonal over each symbol period T.

$\int_{\mathrm{mT}}^{(m+1)T} c_i\left(t\right)c_j\left(t\right)\mathrm{dt} = \begin{cases} 1 & if\ j = i \\ 0 & if\ j \neq i \end{cases}$



For PAM signal,

$x_i\left(t\right) = \sum_k X_k^i p\left(t - kT\right)$

$s^{\text{CDMA}}\left(t\right) = \left[\sum_{i=1}^{K} c_i\left(t\right) x_i\left(t\right)\right] \cos(2\pi f_c t)$

Assuming no noise,

$\int \left(\sum_{i=1}^{K} c_i\left(t\right) x_i\left(t\right)\right) c_j\left(t\right) dt = x_j\left(t\right)$